INSTRUCTIONS

1. This has to be your own work. Cheating on the exam is not tolerated, and will fetch you a zero for the test.

2. TIME LIMIT: 90 minutes

3. There are 5 pages and 4 questions on the exam. Each question appears on a different page. Read each question carefully.

4. The exam is worth 105 points including 5 extra credit points. The distribution of these points is clearly indicated on the exam.

5. Solve each problem in sufficient detail in the space provided. Please use both sides of each page as needed.

6. Write clearly, including all the steps to the final solution. If I can’t read it, you won’t get credit.

7. This is a closed book exam. You may use two crib sheets of formulas on the exam.

8. You can also use an electronic calculator on the exam.
1. [30 points] Consider the initial value problem

\[
\frac{dy}{dx} = x^2(1+y), \quad y(0) = 3.
\]  

(a) [15 points] Solve (1) as a separable differential equation for \( y(x) \).

(b) [15 points] Solve (1) as a linear differential equation for \( y(x) \).

You should get the same expression for \( y(x) \) in either case.

\[
\frac{dy}{dx} = x^2(1+y) \quad \Rightarrow \quad \int \frac{dy}{1+y} = \int x^2 \, dx + c
\]

\[
\ln |1+y| = \frac{x^3}{3} + c \quad \Rightarrow \quad (1+y) = k e^{x^3/3}
\]

When \( x=0 \), \( y=3 \):

\[
1 + y = 4 e^{x^3/3} - 1
\]

\[
y = 4 e^{x^3/3} - 1
\]

1(b) Write ODE as

\[
\frac{dy}{dx} + p(x)y = a(x)
\]

where \( p(x) = -x^2 \) and \( a(x) = x^2 \).

(i) \( \mu(x) = e^{\int p(x) \, dx} = e^{-x^3/3} \)

(ii) \[
y(x) = \frac{1}{\mu(x)} \left[ \int \mu(x) a(x) \, dx + c \right]
\]

\[
y(x) = e^{x^3/3} \left[ \int x^2 e^{-x^3/3} \, dx + c \right]
\]

\[
y(x) = e^{x^3/3} \left[ -x^3 e^{-x^3/3} + c \right] = \frac{x^3}{3} + C e^{x^3/3}
\]

\[
y(x) = 1 + C e^{x^3/3}
\]
\[ y(x) = -1 + Ce^{x^3/3} \]

When \( x = 0 \), \( y = 3 \)

\[ C = 4 \]

\[ y(x) = 4e^{x^3/3} - 1 \]

which agrees with your expression from part (a).
2. [25 points] Consider the initial value problem

\[
\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 0, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 1.
\]  \hfill (2)

(a) [20 points] Solve the initial value problem (2) for \( y(t) \).

(b) [5 points] Predict the behavior of the system as \( t \to \infty \), i.e., compute \( \lim_{t \to \infty} y(t) \).

2(a) \quad \text{Auxiliary Equation Is}

\[\lambda^2 - 1 - 6 = 0\]

\[(\lambda - 3)(\lambda + 2) = 0\]

\[\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = -2\]

\[\Rightarrow \quad \text{We have}\]

\[y_1(t) = e^{3t}\]

\[y_2(t) = e^{-2t}\]

\[\text{General Solution Is}\]

\[y(t) = c_1 y_1(t) + c_2 y_2(t)\]

\[= c_1 e^{3t} + c_2 e^{-2t}\]

\[= \begin{cases} 
\text{when } t = 0 \quad y = 1 \\
\Rightarrow \quad c_1 + c_2 = 1 \\
\text{when } t = 0 \quad \frac{dy}{dt} = 1 \\
\Rightarrow 3c_1 - 2c_2 = 1 
\end{cases}\]

\[\text{when } t = 0 \quad \lim_{t \to \infty} y(t) = \frac{3}{5} e^{3t} + \frac{2}{5} e^{-2t}\]

\[\Rightarrow \quad \lim_{t \to \infty} y(t) = \infty\]
3. [25 points] An object of mass 5 kg is released from rest 1000 m above the ground and allowed to fall under the influence of gravity. We assume that the force due to air resistance is proportional to the velocity of the object with proportionality constant \( b = 50 \) N·sec/m. Also, \( g = 9.81 \) m/sec\(^2\).

(a) [15 points] Determine the equation of motion of the object.
(b) [10 points] When does the object hit the ground?

3(a) We have

\[
V(t) = \frac{mg}{b} + \left( \frac{v_0 - mg}{b} \right) e^{-bt/m}
\]

and

\[
X(t) = \frac{mg}{b} \cdot t + \frac{m}{b} \left( \frac{v_0 - mg}{b} \right) \left( 1 - e^{-bt/m} \right) \quad \text{(EQUATION OF MOTION)}
\]

\[
m = 5 \quad v_0 = 0 \quad g = 9.81 \quad b = 50
\]

\[
X(t) = \frac{(5)(9.81)}{(50)} t + \frac{5}{50} \left( 0 - \frac{(5)(9.81)}{(50)} \right) \left( 1 - e^{-50t/5} \right)
\]

\[
X(t) = 0.981t - 0.0981 \left( 1 - e^{-10t} \right)
\]

3(b) The object hits the ground when \( X(t) = 1000 \)

\[
1000 = 0.981t - 0.0981 + 0.0981e^{-10t}
\]

Neglecting the \( e^{-10t} \) term, we have

\[
1000 = 0.981t - 0.0981
\]

\[
t = 1019.46 \text{ sec}
\]
4. [25 points] Consider the initial value problem

$$\frac{dy}{dx} = \frac{1}{x^2} - \frac{y - y^2}{x}, \quad y(1) = -1.$$  \hspace{1cm} (3)

(a) [10 points] Verify that $y = -\frac{1}{x}$ is a solution to (3).

(b) [15 points] Use Euler’s method with step size $h = 0.5$ to find an approximate solution to (3) at $x = 2$. Compare this with the exact value of $y$ at $x = 2$ that you obtained in part (a).

\[4(a) \quad \frac{dy}{dx} = \frac{1}{x^2} - \frac{y - y^2}{x}, \quad y(1) = -1\]

\[y = -\frac{1}{x} \quad \therefore \quad \frac{dy}{dx} = -\frac{1}{x^2}\]

ODE is \[\frac{dy}{dx} + \frac{y}{x} + y^2 - \frac{1}{x^2} = 0\]

Plugging in the values of $y$ and $\frac{dy}{dx}$, we have

LHS = \[\frac{1}{x^2} - 1 + \frac{1}{x} - \frac{1}{x^2} = 0 = \text{RHS}\]

\[\text{ODE}\]

\[\therefore \ y = -\frac{1}{x} \ 	ext{satisfies the ODE}\]

Moreover, $y(1) = -1$, i.e., it also satisfies the initial condition

\[\therefore \ y = -\frac{1}{x} \ 	ext{is a solution to (3)}\]

\[4(b) \quad \text{WRITE ODE as}\]

\[\frac{dy}{dx} = f(x, y) \quad \text{,} \quad y(x_0) = y_0\]

\text{where} \quad f(x, y) = \left(\frac{1}{x^2} - \frac{y}{x} - y^2\right), \quad x_0 = 1, \quad \text{and} \quad y_0 = -1\]
**Euler's Method**

(a) \( x_1 = x_0 + h \)
\[ = 1 + 0.5 = 1.5 \]

\( y_1 = y_0 + h f(x_0, y_0) \)
\[ = -1 + 0.5 \left( 1 + 1 - 1 \right) \]
\[ = -1 + 0.5 = -0.5 \]

(b) \( x_2 = x_1 + h \)
\[ = 1.5 + 0.5 = 2 \]

\( y_2 = y_1 + h f(x_1, y_1) \)
\[ = -0.5 + 0.5 \left( \frac{1}{(1.5)^2} - \frac{(-0.5)^2}{1.5} \right) \]
\[ = -0.5 + 0.5 \left( \frac{1}{2.25} - \frac{0.25}{1.5} \right) \]
\[ = -0.2361 \]

Approximate solution
at \( x = 2 \) is \( y_2 = -0.2361 \)

Exact solution
at \( x = 2 \)

is \( y_{\text{exact}} = -\frac{1}{x} = -\frac{1}{2} \)
\[ = -0.5 \]

We can get a better approximate solution at \( x = 2 \) by choosing a smaller step size \( h \).