Homework # 3
Solutions by Kartik

Problem 3)
Exercise 4.3

(a) Consider
\[ y'' + 16y = 0; \quad y(0) = 2, \quad y'(0) = 2 \]

Comparing the diff equation
with \( my'' + by' + ky = 0 \)
we find \( m = 1 \), \( k = 16 \), and \( b = 0 \).

Since there is no damping in the system, the spring mass system will perform simple harmonic motion.

Characteristic equation \( \lambda^2 + 16 = 0 \)

\[ \lambda = \pm 4i \]

\[ y(t) = ce^{4t} + q \sin(4t) \]

\[ x = 0, \quad \beta = 4 \]
Substituting the initial value conditions:

\[ y(0) = 2 \quad \text{we have} \]

\[ y(0) = C_1 = 2 \]

\[ y(t) = C_1 \cos(4t) + C_2 \sin(4t) \]

\[ y'(t) = -4C_1 \sin(4t) + 4C_2 \cos(4t) \]

Since \( y'(0) = 0 \), we have:

\[ y'(0) = 4C_2 = 0 \]

\[ C_2 = 0 \]

\[ y(t) = C_1 \cos(4t) \]

\[ y(t) = 2 \cos(4t) \]

The system performs simple harmonic motion whose amplitude varies between ±2.

Period of oscillation:

\[ T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2} \]
(b) \( y'' + 100y' + y = 0 \), \( y(0) = 1 \), \( y'(0) = 0 \)

\( m = 1, \quad b = 100, \quad k = 1 \)

Since the damping constant \( b = 100 \), we expect \( y(t) \to 0 \) as \( t \to \infty \).

Char. equation is

\( l^2 + 100l + 1 = 0 \)

\( l_{1,2} = \frac{-100 \pm \sqrt{100^2 - 4}}{2} \)

\( l_1 = \frac{-100 + \sqrt{100^2 - 4}}{2} < 0 \)

and \( l_2 = \frac{-100 - \sqrt{100^2 - 4}}{2} < 0 \)

\( y(t) = e^{l_1t} + e^{l_2t} \)

Showing that \( y(t) \to 0 \) as \( t \to \infty \)

There is no need to evaluate the constants \( c_1 \) & \( c_2 \) since we are interested in the long term behavior of the system.
(c) \[ y'' - 6y' + 8y = 0; \quad y(0) = 1; \quad y'(0) = 0 \]

Since the damping constant \( b = -6 \) (NEGATIVE), we expect the system to oscillate with ever increasing amplitude as \( t \to \infty \)

\[ y(t) \to \pm \infty \quad \text{(depending on the signs of } c_1 \text{ and } c_2 \text{.)} \]

Char equation is

\[ \lambda^2 - 6\lambda + 8 = 0 \]
\[ (\lambda - 4)(\lambda - 2) = 0 \]

\[ \lambda_1 = 4; \quad \lambda_2 = 2 \]

\[ y(t) = c_1 e^{4t} + c_2 e^{2t} \]

\[ y(0) = c_1 + c_2 = 1 \]

\[ y'(t) = 4c_1 e^{4t} + 2c_2 e^{2t} \]

\[ \begin{align*}
 2c_1 + (1-c_1) & = 0 \\
 2c_1 + c_2 & = 0
\end{align*} \]

\[ \begin{align*}
 2c_1 + c_2 & = 0 \quad \Rightarrow c_2 = (1-c_1) \\
 c_1 + 1 & = 0 \quad \Rightarrow c_1 = -1 \\
 c_1 + c_2 & = 0 \quad \Rightarrow c_2 = (1-c_1) = 2
\end{align*} \]
\[ y(t) = c_1 e^{4t} + c_2 e^{2t} \]
\[ = -e^{4t} + 2e^{2t} \]
\[ = e^{4t}(-1 + 2e^{-2t}) \]

\[ \lim_{t \to \infty} y(t) = \lim_{t \to \infty} e^{4t}(-1 + 2e^{-2t}) \]
\[ = \left( \lim_{t \to \infty} e^{4t} \right) \left( -1 + 2 \lim_{t \to \infty} e^{-2t} \right) \]
\[ = (\infty)(-1) = -\infty \]

\[ y(t) \to -\infty \text{ as } t \to \infty \]

\((d)\) \[ y'' + 2y' - 3y = 0 \]
\[ y(0) = -2 \quad y'(0) = 0 \]

Since \(k = -3\) is negative, we expect the spring to repel the mass with a force \(F = -ky\) that gets larger as \(y(t)\) increases.

\[ y(t) \to \pm \infty \text{ as } t \to \infty \]

Depending on constants \(c_1\) and \(c_2\).
Chae equation is
\[ x^2 + 2x - 3 = 0 \]
\[ x^2 + 3x - 1 - 3 = 0 \]
\[ x(x+3)-1(x+3) = 0 \]
\[ (x-1)(x+3) = 0 \]

\[ x_1 = 1 \text{ and } x_2 = -3 \]

\[ y(t) = c_1 e^x + c_2 e^{-3x} \]
\[ y(0) = c_1 + c_2 = -2 \]
\[ y'(t) = c_1 e^x - 3c_2 e^{-3x} \]
\[ y'(0) = c_1 - 3c_2 = 0 \]

\[ c_1 = 3c_2 \]
\[ c_1 + c_2 = -2 \]
\[ 3c_2 + c_2 = -2 \]
\[ 4c_2 = -2 \]
\[ c_2 = -\frac{1}{2} \]
\[ c_1 = -3c_2 = \frac{-3}{2} \]

\[ y(t) \rightarrow -\frac{1}{2} e^{-3t} \]

As \( t \rightarrow \infty \)
\[ y(t) \rightarrow -\frac{3}{2} (\infty) \rightarrow -\infty \]
(c) \[ y'' - y' - 6y = 0 \]
\[ y(0) = 1, \quad y'(0) = 1 \]

Again \( b = -1 \) and \( k = -1 \)

we expect the oscillations of ever increasing amplitude (NEGATIVE DAMPING)

and the spring to expect the mass (with a force

\[ F = -ky \] that increases with the displacement \( y \))

\[ y(t) \rightarrow \pm \infty \quad \text{as} \quad t \rightarrow \infty \]

depending on the signs of \( c_1 \) and \( c_2 \).

Characteristic equation is

\[ \lambda^2 - \lambda - 6 = 0 \]

\[ \lambda^2 - 3\lambda + 2\lambda - 6 = 0 \]

\[ \lambda(\lambda - 3) + 2(\lambda - 3) = 0 \]

\[ (\lambda + 2)(\lambda - 3) = 0 \]

\[ \lambda_1 = 3 \quad \text{and} \quad \lambda_2 = -2 \]
\[ y(t) = c_1 e^{3t} + c_2 e^{-2t} \]

\[ y(0) = c_1 + c_2 = 1 \]

\[ y'(t) = 3c_1 e^{3t} - 2c_2 e^{-2t} \]

\[ y'(0) = 3c_1 - 2c_2 = 1 \]

Since \( c_1 + c_2 = 1 \)

\[ c_2 = 1 - c_1 \]

\[ 3c_1 - 2c_2 = 3c_1 - 2(1 - c_1) \]

\[ = 3c_1 - 2 + 2c_1 = 1 \]

\[ 5c_1 = 3 \]

\[ c_1 = 3/5 \]

\[ c_2 = 1 - c_1 = (1 - 3/5) = 2/5 \]

\[ y(t) = \frac{3}{5} e^{3t} + \frac{2}{5} e^{-2t} \]

\[ \lim_{t \to \infty} y(t) = \frac{3}{5} (\infty) + \frac{2}{5} (0) = \infty \Rightarrow y(t) \to \infty, \quad \text{as} \quad t \to \infty \]
Problem 36
Exercise 4.3

Consider
\[ y'' + 2y' + 2y = 0 \quad y(0) = 2 \quad y'(0) = -1 \]

Characteristic equation is \( s^2 + 2s + 2 = 0 \)

\[ s_1, s_2 = -1 \pm \sqrt{4 - 2(4)} \]

\[ = -1 \pm i \]

\[ \alpha = -1, \quad \beta = 1 \]

Using the form (21) for the solution \( y(t) \) we have

\[ y(t) = c_1 e^{(\alpha + i\beta)t} + c_2 e^{(\alpha - i\beta)t} \]

\[ = c_1 e^{(\alpha + i\beta)t} + c_2 e^{(\alpha - i\beta)t} \]

\[ = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t \]

\[ = (c_1 + c_2) e^{-t} \cos t + i(c_1 - c_2) e^{-t} \sin t \]
\[ y(t) = (d_1 + d_2) e^t \cos t + i(d_1 - d_2) e^t \sin t \]

Now \( y = 2 \) when \( t = 0 \)

\[ y(0) = (d_1 + d_2) = 2 \]

\[ d_1 + d_2 = 2 \rightarrow 0 \]

Also \( y'(0) = 1 \)

Now \( y'(t) = (d_1 + d_2) \left( e^t \cos t + e^t \sin t \right) + i(d_1 - d_2) \left( e^t \sin t + e^t \cos t \right) \)

\[ y'(0) = (d_1 + d_2)(1) + i(d_1 - d_2)(1) = 1 \]

\[ (d_1 + d_2) + i(d_1 - d_2) = 1 \rightarrow 0 \]

Substituting 1 in 2 gives
\[ i (d_1 - d_2) = 3 \]

\[
(d_1 - d_2) = -3i \quad \rightarrow (3)
\]

Adding (1) and (3) gives:

\[ 2d_1 = (2 - 3i) \]

\[ d_1 = 1 - \frac{3}{2}i \]

\[ d_2 = (2 - d_1) = \left( 2 - 1 + \frac{3}{2}i \right) \]

\[ = \left( \frac{1 + \frac{3}{2}i}{2} \right) \]

(Note: \( d_1 \) and \( d_2 \) are complex conjugates of each other)

1. \[ y(t) = (d_1 + d_2) e^{-t} \cos t \]

\[ + i/(d_1 - d_2) e^{-t} \sin t \]

\[ y(t) = 2e^{-t} \cos t + 3e^{-t} \sin t \]
Using formula (9)

\[ y(t) = c_1 e^{at} \cos bt + c_2 e^{at} \sin bt \]

\[ = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t \]

\[ y(0) = c_1 = 2 \]

\[ y'(t) = c_1 (-e^{-t} \cos t - e^{-t} \sin t) \]

\[ + c_2 (-e^{-t} \sin t + e^{-t} \cos t) \]

\[ y'(0) = c_1 (-1) + c_2 (1) = 1 \]

\[ c_2 = 3 \]

The queue

\[ y(t) = 2e^{-t} \cos t + 3e^{-t} \sin t \]

which is identical to the expression for \( y(1) \) using formula (21).
Let \( d_1 = A + iB \)
\( d_2 = C + iD \)

\[
y(t) = d_1 e^{(A+\beta)t} + d_2 e^{(A-\beta)t} \rightarrow (21)
\]

\[
= (A + iB)(\cos \beta t + i \sin \beta t) e^{At}
+ (C + iD)(\cos \beta t - i \sin \beta t) e^{At}
\]

(Substituting the values of \( d_1 \) and \( d_2 \) in (21) and using Euler's formula)

\[
y(t) = e^{At} \left( (A + iB)(\cos \beta t + i \sin \beta t) + (C + iD)(\cos \beta t - i \sin \beta t) \right)
\]

\[
= e^{At} \left( A \cos \beta t - B \sin \beta t + C \cos \beta t + D \sin \beta t \right) + i \left( B \cos \beta t + A \sin \beta t + D \cos \beta t - C \sin \beta t \right)
\]

\[
y(t) = y(t) + O i \quad (\text{Since } y(t) \text{ is a REAL no.)}
\]

\[
y(t) = e^{At} \left( (A+C) \cos \beta t + (D-B) \sin \beta t \right) + i e^{At} \left( (A-C) \sin \beta t + (B+D) \cos \beta t \right)
\]
\[ y(t) = \text{e}^{at} (A + iB) \text{ and } x_2 = (c + i\beta) \]

\[ d_1 = (A + iB) \text{ and } d_2 = (c + i\beta) \]

Conjugates of each other.

A = C and B = -D

A - C = 0 and B + D = 0

(A - C) \sin B + (B + D) \cos B + (A - C) \sin B + (B + D) \cos B + 0 = 0

Since expression 2 is true, we must have

\[ \text{e}^x \forall x \in \mathbb{R} \]
Show that
\[ y_1(t) = e^{xt} \cos \beta t \quad \text{and} \quad y_2(t) = e^{xt} \sin \beta t \]
are linearly independent.

We will show this by showing that equation (11) on page 163 does not hold for any real number \( \tau \).

Now
\[ y_1(t) = e^{xt} \cos \beta t \]
\[ y_1'(t) = x e^{xt} \cos \beta t - \beta e^{xt} \sin \beta t \]
\[ y_2(t) = e^{xt} \sin \beta t \]
\[ y_2'(t) = x e^{xt} \sin \beta t + \beta e^{xt} \cos \beta t \]
\[ y_1(t) y_2'(t) - y_1'(t) y_2(t) = (e^{xt} \cos \beta t) (\beta e^{xt} \sin \beta t + e^{xt} \cos \beta t) - (x e^{xt} \cos \beta t - \beta e^{xt} \sin \beta t) (e^{xt} \sin \beta t) \]
\[ = \beta e^{2xt} \cos \beta t \sin \beta t + \beta e^{2xt} \cos^2 \beta t - \beta e^{2xt} \cos \beta t \sin \beta t \]
\[ = \beta e^{2xt} \cos \beta t (\cos^2 \beta t + \sin^2 \beta t) = 2 \beta e^{2xt} \]
\[ y(x) = C e^{3x} + C_2 e^{-2x} \]

\[ \lambda_1 = 3 \text{ and } \lambda_2 = 2 \]

\[ \lambda_1 (x-3) (x-2) = 0 \]

\[ x^2 - 5x + 6 = 0 \]

Exercise 4.4

Problem 15

Note: If \( \beta = 0 \), then \( y(0) = e^{\lambda x} \) and \( y(\lambda) \) are linearly independent

Chase equation is

\[ \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6 y = x e^x \]

\[ \lambda \neq 0 \text{ since } x = 2 \text{ are imaginary} \]

\[ \lambda \neq 0 \text{ because roots would be real.} \]

If \( \beta = 0 \) the roots would be real.

\[ y_1 (x) = e^{2x} \]

\[ y_2 (x) = e^{-x} \]

\[ y_1 (x) y_2 (x) - y_1 (x) y_2 (x) \]

\[ \beta = 0 \]

\[ x \neq 0 \text{ since } \lambda \neq 0 \]
Since the LHS contains only polynomial and exponential terms, we use the 1st formula in the table.

We have

\[ y_p(x) = (A_1 x + A_0) e^x \]

Also, \( x = 1 \) is not a root of characteristic equation \( s = 0 \).

\[ y_p(x) = (A_1 x + A_0) e^x \]

\[ y_p'(x) = A_1 e^x + (A_1 x + A_0) e^x \]

Also,

\[ y_p''(x) = A_1 e^x + A_1 e^x + (A_1 x + A_0) e^x \]

\[ = 2 A_1 e^x + (A_1 x + A_0) e^x \]

\[ y_p''(x) - 5 y_p'(x) + 6 y_p(x) \]

\[ = (2 A_1 e^x + (A_1 x + A_0) e^x) \]

\[ - 5 \left( A_1 e^x + (A_1 x + A_0) e^x \right) + 6 \left( A_1 x + A_0 \right) e^x \]

\[ = -3 A_1 e^x + 2 \left( A_1 x + A_0 \right) e^x \]
\[
\begin{align*}
yp''(x) - 5yp'(x) + 6yp(x) &= -3A_1e^x + 2(A_1x + A_0)e^x \\
&= -3A_1e^x + 2A_0e^x + 2A_1xe^x \\
&= xe^x \\
(-3A_1 + 2A_0)e^x + 2A_1xe^x &= xe^x \\
\therefore 2A_1 &= 1, \quad A_1 = \frac{1}{2} \\
-3A_1 + 2A_0 &= 0 \\
\therefore 2A_0 &= 3A_1 = 3/2 \\
\therefore A_0 &= \frac{3}{4} \\
yp(x) &= (A_1x + A_0)e^x \\
&= \left(\frac{1}{2}x + \frac{3}{4}\right)e^x \\
&= \frac{1}{2}x e^x + \frac{3}{4} e^x
\end{align*}
\]
Exercise 4.4

Problem 31. Consider

\[ y'' + 2y' + 2y = 8 + 3e^{-t} \sin t \]

The characteristic equation is

\[ r^2 + 2r + 2 = 0 \]

\[ r_1, r_2 = -1 \pm i \]

\[ \lambda = -1 \quad \beta = 1 \]

Since the LHS contains a sinusoidal term, use the 2nd formula.

\[ y_p(t) = t^5 \left( A_3 t^3 + A_2 t^2 + A_1 t + A_0 \right) e^{-t} \cos t \]

\[ + t^5 \left( B_3 t^3 + B_2 t^2 + B_1 t + B_0 \right) e^{-t} \sin t \]

Now, since \((-1 + i)\) is also a root of the characteristic equation, \( s = 1 \).

\[ y_p(t) = t \left( A_3 t^3 + A_2 t^2 + A_1 t + A_0 \right) e^{-t} \cos t \]

\[ + t \left( B_3 t^3 + B_2 t^2 + B_1 t + B_0 \right) e^{-t} \sin t \]
Exercise 4.5

Problem 4.3

The 2nd order differential equation describing the spring mass system is

\[ my'' + by' + ky = f(t) \]

where \( F(t) = 5smt \)

\[ y'' + 4y' + 3y = 5smt \]

(Since \( m = 1 \), \( b = 4 \), \( k = 3 \))

Initial conditions are

\[ y(0) = \frac{1}{2}, \quad y'(0) = 0 \]

Characteristic equation is

\[ \lambda^2 + 4\lambda + 3 = 0 \]

\[ (\lambda + 3)(\lambda + 1) = 0 \]

\[ \lambda_1 = -3 \quad \text{and} \quad \lambda_2 = -1 \]

\[ y(t) = c_1 e^{-3t} + c_2 e^{-t} \]

Since the LHS is \( 5smt \), we can assume

\[ y(t) = A \cos t + B \sin t \]
\[ y_p(t) = (A \cos t + B \sin t) \]

\[ y_p''(t) = (-A \cos t - B \sin t) \]

\[ y_p'' + 4y_p' + 3y_p = (A \cos t + B \sin t) \]

\[ + 4 (A \cos t + B \sin t) \]

\[ + 3 (A \cos t + B \sin t) \]

\[ = (2A + 4B) \cos t \]

\[ + (2B - 4A) \sin t \]

\[ = 5 \sin t \]

\[ 2A + 4B = 0 \quad A + 2B = 0 \]

\[ 2B - 4A = 5 \]

\[ 2B - 4(-2B) = 5 \]

\[ 10B = 5 \]

\[ B = \frac{1}{2} \]

\[ A = -2B = -2(\frac{1}{2}) \]

\[ = -1 \]

\[ y_p(t) = - \cos t + \frac{1}{2} \sin t \]
The general solution is
\[ y(t) = y_h(t) + y_p(t) \]
\[ = c_1 e^{-3t} + c_2 e^{-t} - \cos t + \frac{1}{2} \sin t \]

\[
y(0) = c_1 + c_2 - 1 = \frac{1}{2}
\]

\[
\begin{align*}
  c_1 + c_2 &= 3/2 \\
  \therefore c_1 + c_2 &= 3/2 \rightarrow (1)
\end{align*}
\]

\[ y'(t) = -3c_1 e^{-3t} - c_2 e^{-t} + \sin t + \frac{1}{2} \cos t \]

\[ y'(0) = -3c_1 - c_2 + \frac{1}{2} = 0 \]

\[
\begin{align*}
  3c_1 + c_2 &= 1/2 \\
  \therefore 3c_1 + c_2 &= 1/2 \rightarrow (2)
\end{align*}
\]

Subtracting (1) from (2) gives
\[ 2c_1 = -1 \]
\[ c_1 = -1/2 \]

\[ c_1 + c_2 = 3/2 \]

\[ -1/2 + c_2 = 3/2 \]

\[ \therefore c_2 = 2 \]

\[ y(t) = -\frac{1}{2} e^{-3t} + \frac{2e^{-t}}{2} + \frac{3}{2} e^{-t} - \cos t + \frac{1}{2} \sin t \]