Your Name: SOLUTION
For purpose of anonymous grading, please do not write your name on the subsequent pages.

This examination consists of 5 questions, each question counting for the given number of points, adding to a total of 38 points. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult three 8.5in × 11in sheets with notes, but not your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have 120 minutes to do this test.

Problem 1  ____

2  ____

3  ____

4  ____

5  ____

Total  ____
**Problem 1** (7 points, 4 points part a, 3 points part b):

(a) Consider the set of polynomials in $x$ of degree 2 or less with real coefficients, $P_2 = \{ f(x) = a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$. Define the following map

$$\varphi: P_2 \times P_2 \rightarrow \mathbb{R}$$
$$\varphi(f, g) \mapsto f(-1)g(-1) - f(0)g(0) + f(1)g(1)$$

Is $\varphi$ an inner product on $P_2$? Please justify your answer.

NO.

(IP1) is violated for $f(x) = x^2 - 1$: $\varphi(f, f) = 0 \cdot 0 - (-1) \cdot (-1) + 0 \cdot 0 = -1 < 0$.

(b) The map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x/\sqrt{2} + y/\sqrt{2} \\ -x/\sqrt{2} + y/\sqrt{2} \end{bmatrix}$ is a two-dimensional rotation around the origin. By what degree?

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix},$$

hence the rotation is by $-45^\circ$ ($315^\circ$, $-\pi/4$, $7\pi/4$).
Problem 2 (7 points, 4 points part a, 3 points part b):

(a) A person decides to predict the run-time of a program by designing a model as a formula depending only on the size of the input with unknown numerical coefficients and then determining the values of the coefficients by a least squares fit of a number of test run times. Of what pitfalls of this method should the person beware?

- Try as many test runs as possible: this makes the model more accurate.
- Watch for test inputs that yield atypical behaviour. E.g., when modeling a sorting method, testing an almost sorted array may yield test timings that are too low.
- Justify the model by algorithm analysis. One cannot accurately model an \( n \log n \) algorithm with a quadratic model.

(b) Consider the linear differential equation \( y'(t) = a_1 y(t) \), whose solution is \( y(t) = c_1 e^{a_1 t} \), where \( a_1 \) and \( c_1 \) are real constants. Please derive the solution from an eigenvalue problem. Please show all your work.

\[
y'_1 = a_1 y_1; \quad A = [a_1]; \quad \det(\lambda I_1 - A) = \lambda - a_1; \quad \lambda_1 = a_1; \quad \text{nullspace}(\lambda_1 I_1 - A) = \text{nullspace}([0]) = \text{Span}([1]); \quad y_1(t) = c_1 \cdot 1 \cdot e^{a_1 t}.
\]
Problem 3 (6 points): The following is a simplified scenario of an actual application of mathematical proofs to the Internet. Suppose a client computer first downloads data for a square matrix $A$ that allows the rapid computation of $A^{-1}b$, where many $b$’s are subsequently obtained and processed by the client software. The agreed protocol is that the server computer presents a QR factorization of $A$, namely a matrix $Q$ with pairwise orthogonal columns, a diagonal matrix $D$ whose diagonal entries are the reciprocals of the squares of the lengths of the column vectors of $Q$, and an upper triangular matrix $R$ with 1’s on the diagonal such that $A = QR$. The program in the client computer uses $A^{-1}b = R^{-1}(D(Q^Tb))$ where multiplication by $R^{-1}$ is implemented as a back-substitution. However, before installing the data for $Q, D, R$ the client computer must first prove that the transmitted data represent the QR factorization for a non-singular matrix, as anything else may lead to breakdown and possibly the introduction of a computer virus. Please describe what mathematical checks the client computer must perform.

- Check that all matrices $Q, D, R$ are square, namely $n \times n$.
- Check that for the matrix $Q$ one has $\sum_{k=1}^{n} Q_{i,k} Q_{k,j} = 0$ for all $1 \leq i < j \leq n$.
- Check for the matrix $D$ that for all $1 \leq j \leq n$ one has $D_{j,j} \neq 0$ and $1/D_{j,j} = \sum_{i=1}^{n} Q_{i,j}^2$. Furthermore $D$ must be diagonal, i.e., $D_{i,j} = 0$ for all $1 \leq i < j \leq n$.
- Check for the matrix $R$ that $R_{i,j} = 0$ for all $1 \leq j < i \leq n$, and that $R_{i,i} = 1$ for all $1 \leq i \leq n$. 


Problem 4 (12 points, 6 points for each part): Consider the range/column-space $V$ of the matrix $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ as a plane in $\mathbb{R}^3$. The map $P: \mathbb{R}^3 \to V$ that computes the orthogonal projection $\hat{b} = P(b)$ of a vector $b \in \mathbb{R}^3$ onto $V$ is a linear map, hence there exists a matrix $A_P \in \mathbb{R}^{3 \times 3}$, which is different from $A$, such that $\hat{b} = A_P b$.

(a) Please compute $A_P$ via the normal equations of the method of least squares. [Hint: Remember that the columns of $A_P$ are the $P(e_i)$, where $e_i$ are the 3-dimensional unit vectors, i.e., vectors that are zero except for the $i$-th component which is 1, for $i = 1, 2, 3$. Furthermore, $P(e_i) = A\hat{x}_i$ where $\hat{x}_i$ are the least square solutions to the problems $A\hat{x}_i \approx e_i$.]

\[
A^T A\hat{x}_i = A^T e_i \\
\begin{align*}
A^T A &= \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \\
(A^T A)^{-1} &= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}. \\
A^T e_1 &= \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \hat{x}_1 = (A^T A)^{-1} A^T e_1 = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1/3 \end{bmatrix}. \\
A^T e_2 &= \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \hat{x}_1 = (A^T A)^{-1} A^T e_2 = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}. \\
A^T e_3 &= \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \hat{x}_1 = (A^T A)^{-1} A^T e_3 = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}. \\
A_P &= \begin{bmatrix} A\hat{x}_1 & A\hat{x}_2 & A\hat{x}_3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix}.
\]
(b) Please compute $A_p$ by first finding the Gram-Schmidt orthogonalization $u_1, u_2$ of the column vectors of $A$ and then plugging into the formula $\hat{b} = (b^T u_1)/(u_1^T u_1) u_1 + (b^T u_2)/(u_2^T u_2) u_2$ with indeterminate entries for $b$.

\[ u_1 = v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \]
\[ v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad u_2 = v_2 - \frac{v_2^T u_1}{u_1^T u_1} u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}. \]
\[ b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \]
\[ \hat{b} = \frac{x+y}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \frac{x/2+y/2+z}{3/2} \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3x - 1/3y + 1/3z \\ -1/3x + 2/3y + 1/3z \\ 1/3x + 1/3y + 2/3z \end{bmatrix} = A_p \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \]
\[ A_p = \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix}. \]
**Problem 5** (6 points): Consider the $4 \times 4$ matrix

\[
A = \begin{bmatrix}
8 & -2 & -8 & 6 \\
5 & -5 & 4 & 5 \\
5 & -11 & 10 & 5 \\
-9 & 3 & 12 & -7
\end{bmatrix}.
\]

You are given the following eigenvectors for $A$:

\[
\begin{bmatrix}
-1 \\
0 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
-2 \\
1 \\
1 \\
3
\end{bmatrix},
\begin{bmatrix}
2 \\
1 \\
1 \\
-3
\end{bmatrix},
\begin{bmatrix}
-4/3 \\
2/3 \\
1 \\
2
\end{bmatrix}.
\]

Please compute (without Maple) the eigenvalues of $A$ and show your computation. You need not establish that the vectors given are actually eigenvectors of $A$.

\[
\begin{bmatrix}
8 & -2 & -8 & 6 \\
5 & -5 & 4 & 5 \\
5 & -11 & 10 & 5 \\
-9 & 3 & 12 & -7
\end{bmatrix}
\begin{bmatrix}
-1 \\
0 \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
-2 \\
? \\
? \\
?
\end{bmatrix}; \quad \lambda_1 = 2.
\]

\[
\begin{bmatrix}
8 & -2 & -8 & 6 \\
5 & -5 & 4 & 5 \\
5 & -11 & 10 & 5 \\
-9 & 3 & 12 & -7
\end{bmatrix}
\begin{bmatrix}
2 \\
1 \\
1 \\
-3
\end{bmatrix} = \begin{bmatrix}
? \\
-6 \\
? \\
?
\end{bmatrix}; \quad \lambda_3 = -6.
\]

\[
\begin{bmatrix}
8 & -2 & -8 & 6 \\
5 & -5 & 4 & 5 \\
5 & -11 & 10 & 5 \\
-9 & 3 & 12 & -7
\end{bmatrix}
\begin{bmatrix}
-4/3 \\
2/3 \\
1 \\
2
\end{bmatrix} = \begin{bmatrix}
? \\
? \\
? \\
12
\end{bmatrix}; \quad \lambda_4 = 6.
\]