Your Name: ________________________
For purpose of anonymous grading, please do not write your name on the subsequent pages.

This examination consists of 5 questions, each question counting for the given number of points, adding to a total of 21 points. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult two 8.5’ × 11’ sheets with notes, but not your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have 75 minutes to do this test.

Good luck!

Problem 1 _____
  2 _____
  3 _____
  4 _____
  5 _____

Total _____

If you are taking the exam later, please sign the following statement:

I, ____________________, affirm that I have no knowledge of the contents of this exam.

__________________________
Signature
Problem 1 (7 points): Please answer the following questions about vector spaces and matrices. Please, also **justify your answers** briefly.

(a, 1.5pts) Consider $\mathbb{R}^2$ with $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -y_1 - y_2 \end{bmatrix}$ for addition and $\alpha \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \end{bmatrix}$ for scalar multiplication. Is $\mathbb{R}^2$ with these operations a vector space?

No. $\oplus$ is not associative; there is no 0 element.

(b, 1.5pts) For $\mathbb{R}[x]$, the set of polynomials in $x$ with real coefficients, consider the following subset:

$$S = \{ f(x) \in \mathbb{R}[x] \mid f(2) = 0 \}.$$  

Is $S$ a subspace of $\mathbb{R}[x]$ with standard polynomial addition and scalar multiplication?

Yes. $(f_1 + f_2)(2) = f_1(2) + f_2(2) = 0$; $(\alpha f_1)(2) = \alpha \cdot f_1(2) = \alpha \cdot 0 = 0$. 
(c, 1pt) Consider the linear system $Ax = b$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Suppose that $\text{rank}(A) = \text{rank}([A \mid b])$. What must the rank of $A$ be equal to so that the system has a unique solution?

\[
\text{rank}(A) = n.
\]

(d, 1pt) Let $A \in \mathbb{R}^{m \times n}$ be of rank $r$. Consider the set of vectors $S = \{Ay \mid y \in \mathbb{R}^n\}$. $S$ forms a subspace of $\mathbb{R}^m$. What is its dimension?

\[
r.
\]

(e, 2pt) Suppose $W$ is a vector space of dimension $n$, $U$ is a subspace of $W$ of dimension $k$, and $V$ is a subspace of $W$ of dimension $l$. Is it possible that simultaneously $k + l > n$ and $U \cap V = \{0\}$.

No. Let $u_1, \ldots, u_k$ be a basis for $U$, $v_1, \ldots, v_l$ be a basis for $V$. Then $u_1, \ldots, u_k, v_1, \ldots, v_l$ must be lin. dep., because $k + l > n$. Thus

\[
\sum c_iu_i + \sum d_jv_j = 0.
\]

Since $u_1, \ldots, u_k$ are lin. indep., some $d_j \neq 0$. Similarly some $c_i \neq 0$. Thus $0 \neq \sum c_iu_i = -\sum d_jv_j \in U \cap V$. 

3
Problem 2 (3 points): Consider the following matrix:

\[
A = \begin{bmatrix}
a & 0 & 0 & f \\
0 & b & 0 & 0 \\
0 & 0 & c & 0 \\
e & 0 & 0 & d
\end{bmatrix}.
\]

As an expression in \(a, b, c, d, e,\) and \(f,\) compute the determinant of \(A.\)

\[abcd - bcef.\]
Problem 3 (3 points): Consider the following matrix and vector:

\[
A = \begin{bmatrix}
2 & -1 & 0 & 0 & 0 \\
1 & 2 & -1 & 0 & 0 \\
0 & 1 & 2 & -1 & 0 \\
0 & 0 & 1 & 2 & -1 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad b = \begin{bmatrix}
5 \\
6 \\
7 \\
8 \\
9
\end{bmatrix}
\]

(a, 1.5pts) As a quotient of two matrix determinants, write down the \((A^{-1})_{4,2}\), that is, the entry in row 4 and column 2 of the inverse of \(A\). Do not compute the explicit values of the determinants, just give the matrices.

\[
\frac{\det\left(\begin{bmatrix}
2 & -1 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 2
\end{bmatrix}\right)}{\det(A)}
\]

(b, 1.5pts) Consider the linear system \(Ax = b\). As a quotient of two matrix determinants, write down \(x_4\), that is the entry in row 4 of the solution vector \(x\). Again, do not expand the determinants.

\[
x_4 = \frac{\det\left(\begin{bmatrix}
2 & -1 & 0 & 5 & 0 \\
1 & 2 & -1 & 6 & 0 \\
0 & 1 & 2 & 7 & 0 \\
0 & 0 & 1 & 8 & -1 \\
0 & 0 & 0 & 9 & 2
\end{bmatrix}\right)}{\det(A)}
\]
Problem 4 (4 points): Consider the ordered bases

\[ B = (1, x, x^2) \]

and

\[ C = (1, x - 1, (x - 1)(x - 2)) \]

of the vector space \( \mathbb{R}[x] \mod x^3 \), the set of polynomials with real coefficients of degree less than 3. Compute the transforming matrix for a basis change \textbf{from} B-coordinates \textbf{to} C-coordinates.

\[ a + bx + cx^2 = \alpha + \beta(x - 1) + \gamma(x - 1)(x - 2) = \alpha - \beta + 2\gamma + (\beta - 3\gamma)x + \gamma x^2. \]

\[
T \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}.
\]

\[
\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}_{T^{-1}} \cdot \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.
\]

\[ T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}. \]
Problem 5 (4 points): Consider the following matrix and vector:

\[
A = \begin{bmatrix}
-1 & -2 & 0 & 1 \\
0 & -3 & 2 & 0 \\
-2 & 5 & -6 & 2 \\
-1 & 1 & -2 & 1
\end{bmatrix}, \quad b = \begin{bmatrix}
-1 \\
7 \\
-23 \\
-8
\end{bmatrix}
\]

Note that

\[
A \cdot \begin{bmatrix}
1 & -2 & -4/3 & 1 \\
0 & 1 & 2/3 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 \\
-2 & 9 & 0 & 0 \\
-1 & 3 & 0 & 0
\end{bmatrix}
\]

Please compute the general solution of the linear system problem \( Ax = b \) by giving one solution vector \( x_0 \in \mathbb{R}^4 \) with \( Ax_0 = b \) and a basis for the right null space of \( A \).

\[
\text{Nullspace}(A) = \text{Span}( \begin{bmatrix} -4/3 \\ 2/3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} ) \quad \text{(last 2 columns of the transforming matrix } U \text{ with } A \cdot U = L, \text{ where } L \text{ is the CEF)}.
\]

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 \\
-2 & 9 & 0 & 0 \\
-1 & 3 & 0 & 0
\end{bmatrix} \cdot \begin{bmatrix}
y_1 \\
y_2 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
-1 \\
7 \\
-23 \\
-8
\end{bmatrix} \quad (\text{solve } Ly = b). \quad y_1 = 1, \ y_2 = -7/3 \text{ and check } -2y_2 + 9y_2 = -23, -y_1 + 3y_2 = -8.
\]

\[
x_0 = Ty = \begin{bmatrix}
1 & -2 & -4/3 & 1 \\
0 & 1 & 2/3 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
y_1 \\
y_2 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
17/3 \\
-7/3 \\
0 \\
0
\end{bmatrix}.
\]