Your Name: SOLUTION
For purpose of anonymous grading, please do not write your name on the subsequent pages.

This examination consists of 3 problems, which are subdivided into 10 questions, where each question counts for the explicitly given number of points, adding to a total of 43 points. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult two 8.5in × 11in sheets with notes, but not your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have 75 minutes to do this test.

Problem 1   ____

       2   ____

       3   ____

Total   ____

If you are taking the exam later, please sign the following statement:

I, ___________________, affirm that I have no knowledge of the contents of this exam.

________________________
Signature
Problem 1 (18 points) Please answer the following questions and give a brief explanation for your answer.

(a, 4 pts) What is the advantage to solving linear systems when the coefficient matrix is (already) factored into LU-form?

*The cost is lower: One forward and one backward substitution $O(n^2)$ vs. Gaussian elimination $O(n^3)$.*

(b, 4 pts) True or false: the number of multiplications performed when computing the determinant of an $n \times n$ matrix by minor expansion grows exponentially in $n$, in the worst case. Please explain.

*True: $n! \geq 2^{n-1}$ without option remember; $n \cdot 2^n$ with option remember.*

(c, 4 pts) Consider the subset of 2-dimensional vectors, $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \cdot y = 0 \right\} \subset \mathbb{R}^2$. Does this subset form a subspace of $\mathbb{R}^2$? Please explain.

*NO. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in $S$, but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin S$.*

(d, 3 pts) Is it possible that a vector space over $\mathbb{R}$ has a basis that has infinitely many elements? Please explain.

*YES. E.g., $\mathbb{R}[x]$. *

(e, 3 pts) True or false: the linear system $Ax = b$ is consistent if and only if the vector $b$ is in the column space of the matrix $A$. Please explain.

*TRUE. $b = x_1 \cdot A_{e,1} + \cdots + x_n A_{e,n} \in \text{col-space}(A)$. 
Problem 2 (10 points): Consider the following linear system given as a matrix equation:

\[
\begin{bmatrix}
1 & -2 & 0 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix}.
\]

(1)

Here \(a, b, c, d\) are parameters.

(a, 4pts) By Cramer’s rule, please write down two matrices so that the solution \(x_1\) of (1) is the quotient of their determinants.

\[
\begin{bmatrix}
a & -2 & 0 & 0 \\
b & 1 & -2 & 0 \\
c & 0 & 1 & -2 \\
d & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & -2 & 0 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(b, 6pts) Please compute the values of these two matrix determinants by minor expansion. From them, determine the value of \(x_1\) (as a function in \(a, b, c, d\)).

*Minor expansion shows that the determinant of a triangular matrix is the product of its diagonal elements. Hence, by minor expansion along the first column we have*

\[
\det\begin{bmatrix}
a & -2 & 0 & 0 \\
b & 1 & -2 & 0 \\
c & 0 & 1 & -2 \\
d & 0 & 0 & 1
\end{bmatrix} = a \cdot \det\begin{bmatrix}
1 & -2 & 0 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{bmatrix} - b \cdot \det\begin{bmatrix}
-2 & 0 & 0 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{bmatrix} - c \cdot \det\begin{bmatrix}
-2 & 0 & 0 \\
1 & -2 & 0 \\
0 & 0 & 1
\end{bmatrix} - d \cdot \det\begin{bmatrix}
-2 & 0 & 0 \\
1 & -2 & 0 \\
0 & 1 & -2
\end{bmatrix} = a + 2b + 4c + 8d.
\]

*and we have*

\[
\det\begin{bmatrix}
1 & -2 & 0 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 1
\end{bmatrix} = 1 \cdot 1 \cdot 1 \cdot 1 = 1.
\]

Therefore, \(x_1 = a + 2b + 4c + 8d\).
Problem 3: Consider the list of four 4-dimensional vectors, \( S = \begin{bmatrix} 2 & 6 & 6 & 4 \\ 0 & 1 & 1 & 0 \\ 1 & -2 & 0 & 3 \end{bmatrix} \). 

(a, 5 pts) Please pare down the list \( S \) to a subset of linearly independent vectors that spans the same vector space.

Performing row reduction to ref on the corresponding matrix yields

\[
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & -2 & 0 & 3
\end{bmatrix} \rightarrow 
\begin{bmatrix}
1 & 0 & 2 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -2 & -2 & 3
\end{bmatrix} 
\]

\[
\rightarrow 
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{bmatrix} \rightarrow 
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

There are pivot elements in columns 1, 2, and 4, hence the first, second and fourth vector form a basis for \( \text{Span}(S) \).

(b, 6 pts) Please find a matrix \( A \in \mathbb{R}^{m \times 4} \) such that \( \text{Span}_{\mathbb{R}}(S) \) is the null space of \( A \).

\( m = 1, \) because \( \dim(\text{Span}(S)) = 3, \) so the space is a hyperplane in \( \mathbb{R}^4 \). The hyperplane satisfies

\[
a x + b y + c z + d w = 0,
\]

where \( a, b, c, d \) are to be determined. Plugging in vectors in the basis from part a, we get:

\[
\begin{align*}
&\text{For } \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} : a + d = 0; \\
&\text{for } \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} : b - 2d = 0; \\
&\text{for } \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} : c + 3d = 0.
\end{align*}
\]

This is a linear system for \( a, b, c, d \). The variable \( d \) is free, hence we may choose \( d = 1 \). Then \( c = -3d = -3, \) \( b = 2d = 2, \) \( a = -d = -1. \) An answer is

\[
A = \begin{bmatrix} -1 & 2 & -3 & 1 \end{bmatrix}.
\]
(c, 4 pts) Please extend the basis for \( \text{Span}_\mathbb{R}(S) \), which you have determined in part a above, to a basis for the full space \( \mathbb{R}^4 \). [Hint: add vectors that do not belong to the null space of \( A \) computed in part b.]

The vector \( x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \) is linearly independent from the other 3, because it does not lie on the hyperplane (in the null space of \( A \)): \( Ax = (-1) \cdot 1 + 2 \cdot 0 + (-3) \cdot 0 + 1 \cdot 0 = -1 \neq 0 \). Therefore the four vectors \( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \) are linearly independent, and thus form a basis for \( \mathbb{R}^4 \).