Your Name: ______________________
For purpose of anonymous grading, please do not write your name on the subsequent pages.

This examination consists of 3 problems, which are subdivided into 11 questions, where each question counts for the explicitly given number of points, adding to a total of 46 points. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult one 8.5in × 11in sheet with notes, but not your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have 75 minutes to do this test. Good luck!

Problem 1

2

3

Total

If you are taking the exam later, please sign the following statement:

I, ______________________, affirm that I have no knowledge of the contents of this exam.

__________________________
Signature
Problem 1 (19 points)

(a, 3 pts) Please name a well-known mathematician who was born in the USA.

(b, 4 pts) In how many multiplications of pairs of integers can one compute \[
\begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix}^{256}
\] ? Please explain.

(c, 4 pts) By laws that the transposition and inversion operations for matrices satisfy please prove that for an invertible matrix \( A \) we have \((A^T)^{-1} = (A^{-1})^T\).

(d, 4 pts) Suppose that in a Maple session the variable \( A \) has been assigned a \( 4 \times 4 \) invertible matrix. If you then execute the Maple commands \texttt{with(linalg): rref(A);} what answer to you expect?

(e, 4 pts) A binary operation on an arbitrary set \( S \) may or may not satisfy the mathematical laws of (i) associativity, (ii) \( S \) has a unit element, (iii) each element in \( S \) has an inverse element, and (iv) commutativity. For the concrete set \( S = \mathbb{R}^{n \times n} \) and matrix multiplication as the binary operation, which of these four laws are satisfied?
Problem 2 (15 points): Consider the following augmented matrix of a system of linear equations, where the first 3 columns correspond to the variables $x, y, z$ and where $a, b,$ and $c$ are real parameters.

\[
\begin{bmatrix}
1 & 1 & -1 & : & 0 \\
-a & 1 & -1 & : & (1 + a)c \\
0 & b & -b & : & c
\end{bmatrix}
\]  

(a, 4pts) For which values of $a, b,$ and $c$ is the matrix (1) in row echelon form? Please give all the conditions.

(b, 6pts) By performing Gaussian elimination on the cases $a = -1$ and $a \neq -1$ separately, determine for which values of $a, b,$ and $c$ the linear system corresponding to the augmented matrix (1) is consistent. For each condition please state the row-echelon form.

(c, 5pts) For each of the conditions discovered in part b, please perform the back-substitution to solve the system.
Problem 3 (12 points): Consider the following matrix $A$ together with its factorization into elementary matrices. Here $\alpha$ is a non-zero real parameter, and $\beta$ is a real parameter.

$$
\begin{bmatrix}
1 & 0 & 0 \\
\beta & 0 & 1 \\
0 & 1/\alpha & 0
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/\alpha
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\beta & 0 & 1
\end{bmatrix}
$$

$E_1 \cdot E_2 \cdot E_3$

This problem computes $A^{-1}$ as $E_3^{-1} \cdot E_2^{-1} \cdot E_1^{-1}$. Please perform the following steps.

(a, 4 pts) Please write $E_1, E_2, E_3$ as elementary matrices $E_I(\ldots)$ or $E_{II}(\ldots)$ or $E_{III}(\ldots)$.

(b, 4 pts) Please write $E_3^{-1}, E_2^{-1}, E_1^{-1}$ as elementary matrices $E_I(\ldots)$ or $E_{II}(\ldots)$ or $E_{III}(\ldots)$.

(c, 4 pts) Please write out $E_1^{-1}$ as a matrix. Then compute the product $E_2^{-1} \cdot E_1^{-1}$ by performing the elementary row operation for $E_2^{-1}$ determined in part b. Then compute the product $E_3^{-1} \cdot (E_2^{-1}E_1^{-1})$, again by performing the elementary row operation for $E_3^{-1}$ of part b.