Chapter 2: Numerical Integration

There are essentially two main approaches to the numerical calculation of

\[ \int f(x) \, dx \]

- Newton-Cote approach: \( f \) is approximated by an easily integrable function (e.g., interpolating polynomial), and then the integral of the approximate function is computed exactly.

- Gaussian approach: the integral is approximated directly

\[ \int f(x) \, dx \approx \sum_{i=0}^{n} \alpha_i \cdot f(x_i) \]

where the weights \( \alpha_i \), \( i = 0, \ldots, n \), and the nodes \( x_i \), \( i = 0, \ldots, n \), are determined to integrate exactly as large a class of functions as possible (for instance).
We start with the second approach.

**Gaussian quadrature**

Example: Let's find a rule under $x_0, x_1,$ and two weights $w_0, w_1,$ so that

$$\int_{-1}^{1} f(x) \, x \, dx = x_0 \cdot f(x_0) + x_1 \cdot f(x_1)$$

integrate as high degree a polynomial as possible.

4 parameters $\Rightarrow$ 4 conditions.

$\Rightarrow$ we require exactness for $f(x) = 1, x, x^2, x^3$

\[
\begin{align*}
\int_{-1}^{1} f(x) \, dx &= \int_{-1}^{1} 1 \, dx = 2 = w_0 + w_1, \\
\int_{-1}^{1} f(x) \, x \, dx &= \int_{-1}^{1} x \, dx = 0 = w_0 x_0 + w_1 x_1, \\
\int_{-1}^{1} f(x) \, x^2 \, dx &= \int_{-1}^{1} x^2 \, dx = \frac{2}{3} = w_0 x_0^2 + w_1 x_1^2, \\
\int_{-1}^{1} f(x) \, x^3 \, dx &= \int_{-1}^{1} x^3 \, dx = 0 = w_0 x_0^3 + w_1 x_1^3
\end{align*}
\]
By symmetry, we guess $x_1 = -x_0$, $x_0 = x_1 = 1$

(1) $x_0 = x_1 = 1$ \[ \frac{1}{\sqrt{3}} = 0.5774... \]

(2) $\int, \quad \int, \quad \int \Rightarrow \quad \frac{2}{3} = 2x_0^2 \Rightarrow \quad x_0 = -\frac{1}{\sqrt{3}}, \quad x_1 = \frac{1}{\sqrt{3}}$

\[ \Rightarrow \quad \int_{-1}^{1} f(x) \, dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \]

Gauss quadrature with 2 nodes.

Note: $\pm \frac{1}{\sqrt{3}}$ are the roots of the Legendre polynomials of degree 2: $x^2 - \frac{1}{3}$!!

Let the error from the quadrature rule be

\[ E_{\text{GQ}} = \int_{-1}^{1} f(x) w(x) \, dx - \sum_{i=0}^{n} \lambda_i f(x_i) \]

(For the sake of completeness, we consider a weight function $w$).
We say the above quadrature has degree of exactness if

\[ E(p) = 0 \quad \forall p \in P_r. \]

Working with \( n + 1 \) distinct nodes \( x_0, \ldots, x_n \) in \([a, b]\) (arbitrarily at this point), we can always get a degree of exactness of at least \( n \).

Indeed, we can always choose the weights \( \alpha_i \)'s such that

\[
\sum_{i=0}^{n} \alpha_i f(x_i) = \int_{-1}^{1} P_i(x) \, w(x) \, dx
\]

where \( P_i \) is the Lagrange interpolating polynomial of \( f \) at the nodes \( x_0, x_1, \ldots, x_n \).

\[
\prod_{j=0}^{n} P_j(x) = \sum_{j=0}^{n} f(x_j) L_j(x) \quad (L_j(x_i) = \delta_{ij})
\]

so we can take

\[
\alpha_i = \int_{-1}^{1} L_i(x) \, w(x) \, dx
\]

As the previous example shows, it is possible to do better than \( n \) if we choose the nodes judiciously (example: \( n = 1 \) but degree of exactness was 3!).
Theorem (Jacobi)

Let \( P(x) = \frac{1}{n!} (x-x_0) \). If

\[
\int_{-1}^{1} L(x) p(x) \, w(x) \, dx = 0 \quad \forall p \in P_{n-1},
\]

then the quadrature \( \int_{-1}^{1} f(x) \, w(x) \, dx = \sum_{i=0}^{n} a_i \cdot f(x_i) \)

with \( a_i = \int_{-1}^{1} L_i(x) \, w(x) \, dx \) has degree of exactness \( n+1 \).

Proof

Let \( f \in P_{n+1} \Rightarrow \exists q \in P_{n-1} \) and \( r \in P_n \)

such that \( f = L q + r \).

Since \( r \in P_n \), we know

\[
\sum_{i=0}^{n} a_i \cdot r(x_i) = \int_{-1}^{1} r(x) \, w(x) \, dx
\]

provided that \( a_i = \int_{-1}^{1} L_i(x) \, w(x) \, dx \),

\[
r = f - L q
\]

\[
\Rightarrow \sum_{i=0}^{n} a_i \cdot r(x_i) = \int_{-1}^{1} f(x) \, w(x) \, dx - \int_{-1}^{1} L q(x) \, w(x) \, dx
\]

\[
= 0
\]

\[
\Rightarrow \int_{-1}^{1} f(x) \, w(x) \, dx = \sum_{i=0}^{n} a_i \cdot f(x_i)
\]
Remark: The above condition is not only sufficient, it is also necessary.

**Corollary**

The maximum degree of exactness $\int f(x) \, dx = \sum \frac{w_i}{d}$ is $2n+1$.

**Proof**

$n+m = 2n+1$ in the critical case $\Rightarrow m = n+1$ is the largest allowable value. Let's take by contradiction $m = n+2$. $(\#)$ has to be satisfied for all $p \in P_{n+1} \Rightarrow$ we can take $p = d^2$, but then

$(\#) \Leftrightarrow \int d^2(x) w(x) \, dx = 0 \Rightarrow d = 0$

which is a contradiction. \(\blacksquare\)

Back to previous example: $w = 1$

We were shooting for the largest possible degree of exactness

$\Rightarrow \int_{\text{deg } n+1}^{} d(x) p(x) \, dx = 0 \quad \forall p \in P_n$
Therefore, it has to be a multiple of the Legendre polynomial of degree \( n+1 \) \( \Rightarrow \) it has the same roots.

\[ L'(x_j) = 0, \quad j = 0, \ldots, n \]

where \( x_j \) are the roots of the Legendre pol. of deg. \( n+1 \).

**Legendre Gauss quadrature**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_i )</th>
<th>( a_i )</th>
<th>deg. of exactness</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 &amp; &amp; ( \pm \frac{1}{\sqrt{3}} ) &amp; 1 &amp; 3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 &amp; &amp; ( 0 ) &amp; ( \frac{1}{3} ) &amp; 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; &amp; ( \pm \frac{1}{\sqrt{15}} ) &amp; ( \frac{1}{5} ) &amp; 9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 &amp; &amp; ( \pm \frac{1}{35} \sqrt{525-70\sqrt{30}} ) &amp; ( \frac{1}{32} (18+\sqrt{30}) ) &amp; 7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; &amp; ( \pm \frac{1}{35} \sqrt{525+70\sqrt{30}} ) &amp; ( \frac{1}{36} (18-\sqrt{30}) ) &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 &amp; &amp; ( 0 ) &amp; ( \frac{1}{320} ) &amp; 9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; &amp; ( \pm \frac{1}{21} \sqrt{245-14\sqrt{70}} ) &amp; ( \frac{1}{320} (322+13\sqrt{70}) ) &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; &amp; ( \pm \frac{1}{21} \sqrt{245+14\sqrt{70}} ) &amp; ( \frac{1}{320} (322-13\sqrt{70}) ) &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
What if the interval is not $[-1, 1]$?

We just map it to $[-1, 1]$

$$\int_a^b f(x) \, dx = ?$$

Let $\psi : [-1, 1] \to [a, b]$

$$x \mapsto \psi = \psi(x) = \frac{b-a}{2} x + \frac{a+b}{2}$$

$$\Rightarrow \int_a^b f(\psi) \, d\psi = \int_1^1 f(\psi(x)) \psi'(x) \, dx = \frac{b-a}{2} \int_1^1 f(\psi(x)) \, dx$$

$$\Rightarrow \sum_{i=0}^{n} \frac{b-a}{2} \Delta_i \int f(\psi(x_i)) \approx \int_a^b f(\psi) \, d\psi$$

Therefore, if $\omega_0, \ldots, \omega_n$ and $x_0, \ldots, x_n$ are the weights and nodes over $[-1, 1]$, the weights and nodes over $[a, b]$ are

$$\omega_i = \frac{b-a}{2} \omega_i, \quad i = 0, \ldots, n$$

$$\chi_i = \frac{b-a}{2} \chi_i + \frac{a+b}{2}, \quad i = 0, \ldots, n$$
Sometimes it is useful to include the endpoints as quadrature nodes.

Gauss–Radau integration includes one of the endpoints as node, say, $-1$.

Instead of considering $L(x) = \sum_{j=0}^{n} (x-x_j) = L_{n+1}(x)$, where $L_{n+1}$ denotes the Legendre polynomial of degree $n+1$, we take

$$\tilde{L}(x) = L_{n+1}(x) + \alpha L_n(x)$$

where $\alpha$ is taken so that $\tilde{L}(-1) = 0$ ($\alpha = -\frac{L_{n+1}}{L_n}$).

So, \eqref{eq:1} reads

$$\int_{-1}^{1} \tilde{L}(x) p(x) \, dx = 0 \quad \forall p \in P_{n-1}$$

$\Rightarrow m = n$

and hence by the Theorem due to Jacobi–Gauss–Radau integration, with the nodes $x_0 = -1, x_1, \ldots, x_n$ being the roots of $\tilde{L}$ and the weights $\tilde{w}_i = \int_{-1}^{1} \tilde{L}(x) \, dx$ with $\tilde{w}(x_j) = \delta_{ij}$, $i,j = 0, \ldots, n$ has deg. of exactness $2$.
(3) Simpson formula \( n=2 \)

\[
\alpha = \frac{b-a}{2} \int_{0}^{2} \frac{(t-1)(t-2)}{(t-0)(t-2)} \, dt = \frac{b-a}{3} \frac{1}{2}
\]

\[
\alpha' = \frac{b-a}{2} \int_{0}^{2} \frac{(t-0)(t-1)}{(t-0)(t-2)} \, dt = \frac{b-a}{3} \frac{4}{2}
\]

\[
\alpha'' = \frac{b-a}{2} \int_{0}^{2} \frac{(t-0)(t-1)}{(t-0)(t-2)} \, dt = \frac{b-a}{3} \frac{1}{2}
\]

\[
\Rightarrow I_2(f) = \frac{b-a}{6} \left( f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)
\]

\[\text{Theorem}\]

if even then

\[
\int_{a}^{b} f(x) \, dx - I_n(f) = \frac{M_n}{b-a} \frac{x^{n+3}}{(n+2)!}
\]

for every \( f \in C^{n+2}([a,b]), \) \( f \in (a,b) \)

with \( M_n = \int_{a}^{b} x \, L(t) \, dt \)
The same can be done if both endpoints are included as nodes, i.e., \( x_0 = -1, \), \( x_1, \ldots, x_n = 1 \).

\( L \) is now taken as \( L(x) = \sum L_{n+1}(k) + \lambda L_n(x) + \beta \), where \( \lambda \) and \( \beta \) are chosen so that \( L(-1) = L(1) = 0 \).

The corresponding quadrature formula has degree of exactness \( 2n-1 \) and is known as Gauss-Lobatto integration (see Project 2).

Similar principles can be applied to find quadrature formulas with other weights (read Section 10.3 on Lobatto's integration) or for some multidimensional domains.

Let us now turn to

**Interpolatory quadratures**

\[ \int_a^b f(x) \, dx \approx \int_a^b T[f](x) \, dx \]

where \( T[f] \) is an easily integrable approximation.
For instance

$$\Pi f(x) = \sum_{i=0}^{\infty} f(x_i) L_i(x)$$

where $$x_0, x_1, \ldots, x_n$$ are $$n+1$$ nodes in $$(a, b)$$

$$\int_a^b f(x) dx \times \sum_{i=0}^{\infty} a_i f(x_i) = \int_0^1 f$$

where $$a_i = \int_a^b L_i(x) dx$$ (again).

Here, though, the nodes will not be optimized.

Examples

1. **Midpoint Formulas**

   $$\Pi f(x) = f(x_0) = f\left(\frac{a+b}{2}\right)$$

   $$(n=0)$$

   $$\Rightarrow \int_0^1 f = (b-a) f\left(\frac{a+b}{2}\right)$$
Before looking at the other example, let's simplify things a little.

\[ x_i = \int_a^b L_i(x) \, dx \]

Let's assume that the nodes form a uniform partition of \([a, b]\)

\[ x_i = a + ih \quad i = 0, \ldots, n \quad h = \frac{b-a}{n} \]

\[ x = a + ht \]

(2) Trapezoidal rule

\[ (n=1) \]

\[ \alpha_0 = (b-a) \int_0^{1} \frac{t-1}{0-1} \, dt = \frac{b-a}{2} \]

\[ \alpha_1 = (b-a) \int_0^{1} \frac{t-0}{1-0} \, dt = \frac{b-a}{2} \]

\[ \Rightarrow \quad I_1(f) = \frac{b-a}{2} \left( \int_a(f) + \int_b(f) \right) \]
Simpson formula \( n = 2 \)

\[
\alpha_0 = \frac{b-a}{2} \int_0^2 \frac{(t-1)(t-2)}{(t-1)(t-2)} \, dt = \frac{b-a}{2} \frac{1}{3}
\]

\[
\alpha_1 = \frac{b-a}{2} \int_0^2 \frac{(t-0)(t-2)}{(t-1)(t-2)} \, dt = \frac{b-a}{2} \frac{4}{3}
\]

\[
\alpha_2 = \frac{b-a}{2} \int_0^2 \frac{(t-0)(t-1)}{(t-2)(t-1)} \, dt = \frac{b-a}{2} \frac{1}{3}
\]

\[
\Rightarrow I_2(f) = \frac{b-a}{8} \left( f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)
\]

Let's rewrite \( I_n(f) = \frac{b-a}{n_5} \sum_{i=0}^{n_5} \alpha_i f(x_i) \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \alpha_i )</th>
<th>( n_5 )</th>
<th>error</th>
<th>name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1 4 1</td>
<td>2</td>
<td>( \frac{h^3}{12} f^{(2)}(\bar{x}) )</td>
<td>Trapezoidal rule</td>
</tr>
<tr>
<td>3</td>
<td>1 3 3 1</td>
<td>6</td>
<td>( \frac{h^5}{90} f^{(4)}(\bar{x}) )</td>
<td>Simpson's rule</td>
</tr>
<tr>
<td>4</td>
<td>7 32 12 32 7</td>
<td>8</td>
<td>( \frac{h^7}{180} f^{(6)}(\bar{x}) )</td>
<td>3/8-rule</td>
</tr>
<tr>
<td>5</td>
<td>19 75 50 50 75 19</td>
<td>90</td>
<td>( \frac{h^9}{945} f^{(8)}(\bar{x}) )</td>
<td>Milne's rule</td>
</tr>
<tr>
<td>6</td>
<td>41 216 27 272 27 216 41</td>
<td>288</td>
<td>( \frac{h^9}{12096} f^{(10)}(\bar{x}) )</td>
<td>Weddle's rule</td>
</tr>
</tbody>
</table>

We'll see how to get the error terms below.
Other integration formulas can be obtained by considering \( \Pi f \)

\[
\int_a^b f(x) \, dx = \int_a^b \Pi f(x) \, dx
\]

where \( \Pi f \) is a Hermite interpolation polynomial.

Assume that \( y_i^k = f^{(k)}(x_i) \) are given \( i = 0, \ldots, n \), \( k = 0, \ldots, m_i \)

Let \( N \) be the total number of data, i.e., \( N = \sum_{i=0}^{n} (m_i + 1) \)

It can be shown that there exists a unique polynomial \( H \in \mathbb{P}_N \) such that

\[
H(x_i) = y_i^k \quad i = 0, \ldots, n, \quad k = 0, \ldots, m_i
\]

\( H \) takes the form

\[
H(x) = \sum_{i=0}^{n} \sum_{k=0}^{m_i} y_i^k \cdot L_{ik}(x),
\]

where the \( L_{ik}(x) \) are generalized Laguerre polynomials which satisfy

\[
\frac{d^p}{dx^p} L_{ik}(x_j) = \begin{cases} 1 & i = j \text{ and } k = p \\ 0 & \text{otherwise} \end{cases}
\]
These polynomials can be constructed as follows. Let
\[ l_{ij}(x) = (x-x_i)^j \prod_{k=0}^{n} \frac{(x-x_k)}{(x_i-x_k)^{k+1}} \]
and let's set \( l_{ij}(x) = l_{w_i}(x) \) for \( i = 0, \ldots, n \). We then define the remaining polynomials recursively as
\[ l_{ij}(x) = l_{ij}(x) - \sum_{k=j+1}^{n} l_{ij}(x_i) l_{ik}(x) \]
for \( j = m_i - 1, m_i - 2, \ldots, 0 \).

Exercise: Check that the \( l_{ij} \)'s do the job.

Example: Assume the function and its derivative are known at \( x_i \):
\[ f^{(k)}(x_i) = y_i \quad i = 0, \ldots, n, \quad k = 0, 1 \]
\( \Rightarrow \) \( w_i = 1, \quad i = 0, \ldots, n \)
\( \Rightarrow N = 2(n+1) \)
\[ L_{ij}(x) = (x-x_i)^j \prod_{k=0}^{n} \frac{(x-x_k)}{(x_i-x_k)^{k+1}} \]
\( \Rightarrow \) \[ L_{ij}(x) = \sum_{k=j+1}^{n} l_{ij}(x_i) l_{ik}(x) \]
\( \Rightarrow \) etc.
We can now turn to error estimates.

Consider the integration error

\[ E(f) = I(f) - \int_a^b f(x) \, dx \]

Note that \( E \) is a linear operator.

**Theorem (Peano)**

Suppose \( E(p) = 0 \) \( \forall p \in P_n \). Then for any \( f \in C^{n+1}([a,b]) \)

\[ E(f) = \int_a^b f'(t) K(t) \, dt \]

where \( K(t) = \frac{1}{n!} \mathcal{E}_n((x-t)_+^n) \)

and \( \mathcal{E}_n((x-t)_+^n) \) denotes the error for \((x-t)_+^n\)

when considered as a function of \( x \).

\( K \) is called the Peano kernel.
Example: Simpson: \( a(b) = (-1, 1) \)

\[ E(f) = \frac{1}{3} f(-1) + \frac{4}{3} f(0) + \frac{1}{3} f(1) - \frac{1}{3} \int_0^1 f(x) \, dx \]

Let \( p \in P_3 \) and let \( q \in P_2 \) such that

\[ p(-1) = q(-1), \quad p(0) = q(0), \quad p(1) = q(1) \]

Finally, let \( s(x) = p(x) - q(x) \). We have

\[ E(p) = E(s) \]

The polynomial \( s \in P_3 \) and has 3 roots \(-1, 0, 1\)

\[ s(x) = \alpha (x^2 - 1) x \]

But then

\[ E(s) = -\alpha \int_{-1}^1 x (x^2 - 1) \, dx = 0 \]

\( \Rightarrow \) Simpson is exact for \( P_3 \). And so, Pade's theorem can be applied for \( n = 3 \)

\[ \Rightarrow k(t) = \frac{1}{6} \mathbb{E} ( (x-t)^3) \]

\[ = \frac{1}{6} \left( \frac{1}{3} (-1-t)^3 + \frac{4}{3} (0-t)^3 + \frac{1}{3} (1-t)^3 \right) - \int_{-1}^1 (x-t)^3 \, dx + \]
We have
\[
\int_{-1}^{1} (x-t)^3 \, dx = \int_{-1}^{1} (x-t)^3 \, dx = \frac{(1-t)^3}{4}
\]
and for \( t \in (-1, 1) \)
\[
(-1-t)^3 = 0 \quad (1-t)^3 = (1-t)^3 \quad (-t)^3 = (-t)^3 \quad \int_{-t}^{t} \, \text{d}t = \int_{-t}^{t} \, \text{d}t
\]
\[
\implies K(t) = \left\{ \begin{array}{ll}
\frac{1}{2} (1-t)^3 (1+3t) & \text{if } 0 \leq t \leq 1 \\
1(1-t) & \text{if } t < 0
\end{array} \right.
\]
\[
(2 \int_{-1}^{1} K(t) \, dt = \frac{1}{40})
\]

Often, quadrature formulas are not applied to the entire interval \([a,b]\), but instead on a collection of subintervals.

**Example:**

![Diagram](image-url)
The original interval \([a, b]\) is divided into \(N\) sub-intervals \(I_i\), \(i = 0, \ldots, N-1\), where

\[
I_i = [y_i, y_{i+1}], \quad y_i = c_i H + a, \quad H = \frac{b-a}{N}, \quad i = 0, \ldots, N-1
\]

Taking the trapezoidal rule as an example, we have

\[
\int_{x_0}^{x_i} f(x) \, dx = \frac{H}{2} \left( f(x_0) + f(x_i) \right)
\]

where \(x_0 = x_0\) and \(x_i = x_i\)

\[
\Rightarrow \int_{x_0}^{b} f(x) \, dx = \frac{H}{2} \left( f(a) + f(b) \right) + \frac{H}{2} \left( f(a+H) + f(a+2H) \right) + \ldots + f(b-H) + f(b/2) \equiv T_i
\]

What about the error?
On each interval $I_i$, the error is

$$E_i : = \frac{b}{2} \left( f(x_i^2) - f(x_i) \right) - \int_{I_i} f(x) \, dx = \frac{h^3}{12} f^{(2)}(\xi_i)$$

Now

$$T(f) - \int_a^b f(x) \, dx = \frac{h^3}{12} \sum_{i=0}^{N-1} f^{(2)}(\xi_i) = \frac{h^2}{12} (b-a) \frac{1}{N} \sum_{i=0}^{N-1} f^{(2)}(\xi_i)$$

Assuming $f \in C^2([a,b])$, if $\xi \in (a,b)$ such that

$$f^{(2)}(\xi) = \frac{1}{h^2} \sum_{i=0}^{N-1} f^{(2)}(\xi_i)$$

$$\Rightarrow T(f) - \int_a^b f(x) \, dx = \frac{b-a}{12} h^2 f^{(2)}(\xi)$$

$\Rightarrow$ order 2 method.

This result can be generalized.

General composite method

$$\int_a^b f(x) \, dx = \sum_{i=0}^{N-1} \int_{i \cdot \Delta}^{(i+1) \cdot \Delta} f(x) \, dx = \sum_{i=0}^{N-1} \int_{I_i} f(x) \, dx$$
where \( P_i^m f \) is the Lagrange interpolating polynomial of degree \( n \) in the interval \( I_i = [y_i, y_{i+1}] \) i.e.

\[
P_i^m f(x_k) = f(x_k) \quad k = 0, \ldots, n
\]

where \( x_k = y_i + k\Delta x \), \( k = 0, \ldots, n \), \( h = \frac{y_{i+1} - y_i}{n} \).

Denoting \( I_{n,n} f = \sum_{i=0}^{n-1} \int_{I_i} P_i^m f(x) \, dx \)

\[
= \sum_{i=0}^{n-1} \sum_{k=0}^{n} \Delta x \cdot f(x_k)
\]

we have

**Theorem**

Let \( f \in C^0([a,b]) \) and assume \( \Delta x_i = \frac{b-a}{i} > 0 \), \( i = 0, \ldots, n \), then

\[
\lim_{n \to \infty} I_{n,n} f = \int_a^b f(x) \, dx \quad \forall n \in \mathbb{Z}^+
\]

Further,

\[
\left| \int_a^b f(x) \, dx - I_{n,n} f \right| \leq 2(b-a) R(f; H)
\]
where

\[ \delta(f, H) = \sup_{x,y} |f(x) - f(y)|, \quad x, y \in [0,1], \quad x \neq y, \quad |x - y| < \delta \]

(modules of continuity)

Pautz

See both p. 385.