CHAPTER I.

Approximation of functions

Goals:

1. Given a function, replace it by simpler functions (polynomials, trig,...)

2. Given a set of values, construct a function resembling the underlying properties of the phenomenon under consideration.

Key argument

Weierstrass approximation theorem

1. If \( f \) is a continuous function on \([a,b]\), and \( \varepsilon > 0 \) is a given positive number, then there exists a polynomial \( p \) satisfying

\[
|f(x) - p(x)| < \varepsilon \quad \forall x \in [a,b].
\]

2. If \( f \) is a continuous function on \([a,b]\), then there exists a sequence of polynomials \( p_n \) such that

\[
\lim_{n \to \infty} p_n = f \quad \text{uniformly on } [a,b]
\]

3. The polynomials are dense in \( C([a,b]) \) (with the topology of the uniform norm).
Proof

Reduction: enough to consider \([a, b] = [0, 1]\).
- \(f(0) = f(1) = 0\)

\[
\begin{align*}
q(x) &= f(x) - f(0) - x(f(1) - f(0)) \\
q(0) &= q(1) = 0 \quad \text{if } q \text{ can be approximated, so can } f.
\end{align*}
\]

If \(f\) is defined to be zero outside \([0, 1]\), then \(f\) is uniformly continuous on the whole line.

Let \(q_n(x) = C_n (1-x^2)^n\),
where \(C_n\) is the normalizing constant

\[
\int_{-1}^{1} q_n(x) \, dx = 1
\]

Now \((1-x^2)^n \geq 1-nx^2\) (consider \(h(x) = (1-x^2)^n - 1+nx^2\)
\(h(0) = 0, \quad h'(x) > 0 \text{ in } [0, 1]\))

\[
\Rightarrow \frac{1}{C_n} = \int_{-1}^{1} (1-x^2)^n \, dx = 2 \int_{0}^{1} (1-x^2)^n \, dx \geq 2 \int_{0}^{\frac{1}{\sqrt{n}}} (1-x^2)^n \, dx
\]

\[
\Rightarrow 2 \int_{0}^{\frac{1}{\sqrt{n}}} (1-nx^2) \, dx = \frac{2}{3\sqrt{n}} > \frac{1}{\sqrt{n}}
\]

\[
\Rightarrow \frac{1}{C_n} > \frac{1}{\sqrt{n}} \Rightarrow C_n < \sqrt{n}
\]
Now, for any $\delta > 0$, we have

\[
q_n(x) \leq \sqrt{n} \left(1 - \delta^2\right)^{\frac{n}{2}} \quad \text{for any } x, \quad \delta \leq |x| < 1.
\]

\[
\Rightarrow q_n(x) \xrightarrow{n \to \infty} 0 \quad \text{uniformly in } [\delta, 1].
\]

Let $P_n(x) = \int_{-\delta}^{x} f(x+t)q_n(t)\,dt \quad 0 \leq x \leq 1$.

By the assumptions on $f$,

\[
P_n(x) = \int_{-\delta}^{1-x} f(x+t)q_n(t)\,dt = \int_{0}^{1-x} f(t)q_n(t-x)\,dt
\]

which is a polynomial in $x$!

Given $\epsilon > 0$, we choose $\delta > 0$ such that

\[
|y-x| < \delta \Rightarrow |f(y) - f(x)| < \frac{\epsilon}{2} \quad (\text{or by uniform continuity of } f).
\]

Define $\delta_0 = \sup \{|f(x)|\}$.

Now, $q_n$ is normalized.

\[
|P_n(x) - f(x)| = \left| \int_{-\delta}^{1-x} f(x+t)q_n(t)\,dt \right|
\]

\[
\leq \int_{-\delta}^{1-x} |f(x+t) - f(x)|q_n(t)\,dt
\]

\[
\leq \int_{-\delta}^{-\delta} + \int_{-\delta}^{1-x} + \int_{1-x}^{1}\delta_0 + \int_{1-x}^{1-x} \delta_0
\]

\[
\leq \int_{-\delta}^{-\delta} + \int_{-\delta}^{1-x} + \int_{1-x}^{1-x} \delta_0
\]

\[
= \delta_0 (1 - \delta)
\]
\[ \leq 2 \int_{-\infty}^{-\delta} q(u) \, du + \frac{3}{2} \int_{-\delta}^{\delta} q(u) \, du + 2 \int_{\delta}^{\infty} q(u) \, du \]
\[ \leq 4 \int_{0}^{\infty} \sin \left( (1-x) \right) \, dx + \frac{3}{2} \]
\[ < 3 \quad \text{for } u \text{ large enough}. \]

It is very instructive to look at the shape of \( q_n \! \); we have constructed polynomial approximations of \( \delta \)-Dirac masses!

This makes sense: \( P_n(x) \approx \int f(t) \delta(t-x) \, dt \)
\[ \approx < \frac{f}{f}, \delta_x > = f(x). \]

Remark: the above construction is not practical at all; it converges very slowly and the calculations can be messy.

Exercise: Compute and plot some of the polynomials for \( f(x) = \sin \pi x \).
Next idea: if \( f \) is smooth, one could consider Taylor polynomials.

Example: \( f(x) = \frac{1}{x} \) in \([\delta, 1]\) \(0 < \delta < 1\).

Let \( P_n(x) \) be the Taylor poly. of degree \( n \) corresponding to the expansion of \( f \) about \( 1 \).

We have
\[
P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} (x-1)^k.
\]

But
\[
f^{(n)}(x) = (-1)^n n! x^{-n-1}
\]

\(\Rightarrow\) \( P_n(x) = \sum_{k=0}^{n} (-1)^k (x-1)^k \).

Exercise: find the radius of convergence of this sequence (see old practice).

With Taylor, all the information is concentrated at one point \(\Rightarrow\) intrinsically local. We need something else.

Let \( x_0 < x_1 < \ldots < x_n \) be a family of \( n+1 \) points in \( \mathbb{R} \), and let's consider the Lagrange polynomials

\[
L_i(x) = \frac{(x-x_0)(x-x_i)(x-x_{i+1}) \ldots (x-x_n)}{(x_i-x_0)(x_i-x)(x_i-x_{i+1}) \ldots (x_i-x_n)}, \quad i = 0, 1, \ldots, n
\]
We clearly have \[ L_i(x) \in \mathbb{P}_n, \quad i = 0, \ldots, n \]

\[ L_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

The above polynomial \( L_i \) is uniquely defined. Indeed, if \( \tilde{L}_i \) is another polynomial with the above properties, then we have

\[ L_i - \tilde{L}_i \in \mathbb{P}_n \]

and \( L_i - \tilde{L}_i \) has \( n + 1 \) different zeros, namely \( x_0, x_1, \ldots, x_n \Rightarrow L_i = \tilde{L}_i \).

**Theorem (Lagrange Interpolation):**

Let \( (x_i, f_i), \quad i = 0, \ldots, n \) \( x_i \neq x_j \) for \( i \neq j \) \( n+1 \) arbitrary support points. Then

\[ P(x) = \sum_{i=0}^{n} f_i \frac{\prod_{k=0,k \neq i}^{n} x-x_k}{\prod_{k=0,k \neq i}^{n} x_i-x_k} \]

is the unique polynomial in \( \mathbb{P}_n \) which interpolates the above data, i.e.

\[ P(x_j) = f_j, \quad j = 0, \ldots, n. \]
\[ P(x) = \sum_{i=0}^{n} f_i \delta_i(x) - \tilde{f} \]

uniqueness: already proved.

To analyze the error in polynomial interpolation, we need the following auxiliary result.

**Generalized Rolle Theorem**

Let \( f \in C([a,b]) \) be \( n+1 \) times differentiable in \( (a,b) \). If \( f \) vanishes at \( n+1 \) distinct points \( x_0, \ldots, x_n \) in \( [a,b] \), then there exists \( \exists \in (a,b) \) with \( f^{(n)}(\exists) = 0 \).

**Proof**

Elementary.

**Theorem**

Let \( x_0, \ldots, x_n \) be \( (n+1) \) distinct points in \( [a,b] \). Then if \( f \in C^{n+1}([a,b]) \), for each \( x \in [a,b] \), there exist \( \exists(x) \) in \( (a,b) \) such that

\[ f(x) - p(x) = \frac{f^{(n+1)}(\exists(x))}{(n+1)!} (x-x_0) \cdots (x-x_n) \]

where \( p \) is the interpolating polynomial defined above.
Remark

Note the relation with the Taylor error formula:

\[ f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \]

All the information is at the same point.

Proof

1st case: \( \frac{1}{n} \int_{x_0}^{x} f'(t) - p'(t)\, dt = f(x) - p(x) \)

2nd case: \( x \neq x_0, k = 0, 1, \ldots, n \).

Then for \( t \in [a, b] \), we define

\[ g(t) = \int_{x_0}^{t} f'(s) - p'(s)\, ds - (f(x) - p(x)) \frac{(t-x_0)}{(x-x_0)} - \frac{(t-x_k)}{(x-x_k)} \]

\( g \in C^{n+1}([a, b]) \)

\( g(t) = f(x_k) - p(x_k) \)

\( \lim_{i \to 0} \frac{x_k - x_i}{x-x_i} = 0 \).
If \( t = x \), then \( g(x) = f(x) - p(x) - (f(x) - p(x)) \cdot 1 = 0 \).

\( g \in C^{n+1}(Ia,b) \) and vanishes at \( n+2 \) points \( x, x_0, \ldots, x_1 \).

\( \Rightarrow \) General Rolle Th. \( \exists \exists g \in (a,b) \) such that

\[
g^{(n+1)}(x) = 0.
\]

\[
o = \frac{d^{(n+1)}(x)}{dx^{n+1}} - p^{(n+1)}(x) - (f(x) - p(x)) \frac{d^{n+1}}{dx^{n+1}} \left[ \frac{u}{11} \right]_{i=0}^{n} \frac{t-x_i}{(x-x_i)}
\]

Now, \( p \in P_n \Rightarrow p^{(n+1)} = 0 \).

On the other hand, \( \frac{u}{11} \left[ \frac{t-x_i}{(x-x_i)} \right]_{i=0}^{n} \) is a polynomial of degree \( n+1 \) in \( t \):

\[
\frac{u}{11} \left[ \frac{t-x_i}{(x-x_i)} \right]_{i=0}^{n} = \frac{t^{n+1}}{11} \left[ \frac{x-x_i}{(x-x_i)} \right]_{i=0}^{n} + \text{lot}
\]

\[
\Rightarrow \frac{d^{n+1}}{dt^{n+1}} \left[ \frac{t-x_i}{(x-x_i)} \right]_{i=0}^{n} = \frac{(n+1)!}{11} \left( \frac{t-x_i}{(x-x_i)} \right)_{i=0}^{n}
\]
\[ \frac{f^{(n+1)}(x)}{(n+1)!} - \frac{f^{(n+1)}(x_i)}{(n+1)!} \]

and hence

\[ f^{(n+1)}(x) - f^{(n+1)}(x_i) = \frac{f^{(n+1)}(x, x_i)}{(n+1)!} (x-x_i) \]

**Important remarks**

1. As seen in Assignment #1, high degree polynomials tend to be unstable, they are however smooth.

2. To stabilize the interpolation process one can try to choose \( x_0, \ldots, x_n \) so as to minimize

\[ \max_x \left| \frac{f^{(n+1)}(x)}{(n+1)!} (x-x_0) - (x-x_n) \right| \]

Since we have no control over \( f(x) \), we might as well consider minimizing

\[ \max_x \left| (x-x_0)(x-x_n) - (x-x_n) \right| \Rightarrow \text{chebyshev pol.} \]

(see ex. 3, Adv #1).
(3) It is possible to develop recursive formulas for interpolating polynomials (Cote Neville's table, p. 40-43, Newton's formula p. 48-49).

(4) Another idea is to give up some smoothness in order to gain some stability by working on subintervals and piecewise pol. approximations.

example (of piecewise lin. interp.)

A table is prepared for \( f(x) = e^x \), \( 0 \leq x \leq 1 \). What should the step size \( h \) be for piecewise linear interpolation to give an absolute error of at most \( 10^{-6} \)?

\( x_j = jh \)
\( \Delta h = \frac{1}{N} \)
By Theorem, we have on \((x_j, x_{j+1})\)

\[
\left| \frac{f(x) - p(x)}{x} \right| = \frac{1}{2} \left| \int_{x}^{x+1} f''(z)(x-z)(x-k_{j})(x-k_{j+1}) \right|
\]

\[
\leq \frac{1}{2} \max_{z \in (x_j, x_{j+1})} e \max_{x \in (x_j, x_{j+1})} |(x-z)(x-k_{j})(x-k_{j+1})| \leq \frac{e}{8} h^2 = \frac{e}{8} \frac{h^2}{4} = \frac{e}{8} h^2
\]

We want \( \left| \frac{f(x) - p(x)}{x} \right| \leq 10^{-6} \)

or \( \frac{e}{8} \frac{h^2}{4} \leq 10^{-6} \Rightarrow h \leq 1.72 \cdot 10^{-3} \)

\( h = .001 \) is OK.

Spline interpolation is also a way to fight against the instabilities of the standard Lagrange interpolation.

It is best understood by analogy with a bending beam.
When looking for a function having the values $y_0$, $y_n$ at the points $x_0$, $x_n$, we will think of a long thin beam to which we want to give the correct height $y_0$, $y_n$. We can think of rings at these points and the beam goes through them. At all other points, the force is zero and the beam is free to choose its own shape.

\[ \text{vertical load } f \]
\[ u : \text{displacement}, \quad \frac{du}{dx^2} : \text{curvature} \]

beam, plate : resist bending, restoring force governed by curvature

cord, membrane : don't care about bending, restoring force comes from resisting stretching.
Boundary conditions

1) Simply supported

\[ \frac{d^2u}{dx^2} = 0 \]
\[ u(0) = 0 \]

2) Clamped

\[ u(0) = 0 \]
\[ \frac{du}{dx}(0) = 0 \]

Equation

\[ \frac{d^2}{dx^2} \left( c \frac{d^2u}{dx^2} \right) = f(x) \quad x \in (0,1) \]

\[ c(x) : \text{bending stiffness.} \]

We assume \( c = 1 \) and "general" boundary condition:

\[ \left\{ \begin{array}{l}
\frac{d^4u}{dx^4}(x) = f(x) \quad x \in (0,1) \\
u(0) = a, \quad \frac{du}{dx}(0) = b, \quad u(1) = c, \quad \frac{du}{dx}(1) = d \\
\end{array} \right. \]

If \( f = 0 \), the beam is still bent due to the forces at its ends. We can find the solution explicitly as a third degree polynomial.
Exercise  Hermite cubic.

Check that
\[ u(x) = a(x-1)^3(2x+1) + b(x-1)^2x + c x^2(3-2x) + d x^2/(x-1) \]

satisfy the above equation with \( f(x) \).

Cubic pieces

\[ u_0 \]
\[ u_1 \]
\[ u_n \]

We are going to piece together Hermite cubics. On each interval, we would need the slopes at the endpoints, which are unknown. They will be determined by requiring the approximate function to be \( C^2 \).

We imagine a beam through the support points.

- at \( x = x_k \) for \( k = 0, \ldots, n \) are free ends.
at $x_k$, $k=0, 1, \ldots, n$, there is a concentrated load of unknown magnitude

$$\frac{d^n u}{dx^n} (x) = \sum_{k=0}^{n} g_k \delta_{x_k} (x)$$

where $\delta_{x_k}$ is the Dirac mass at $x_k$, i.e.

$$\int \delta_{x_k} (x) \, dx = 1 \quad \text{and} \quad \int \varphi(x) \, \delta_{x_k} (x) \, dx = \varphi(x_k)$$

(roughly speaking $\delta_{x_k} (x) = \begin{cases} 0 & \text{if } x \neq x_k \\ \infty & \text{if } x = x_k \end{cases}$)

To the left of $x_0$, the beam is straight, although not necessarily horizontal.

Let's integrate the eq. between $-\infty$ and $x$:

If $x < x_0$:

$$\int_{-\infty}^{x} \frac{d^n u}{dx^n} (y) \, dy = u^{(n)} (x) = 0.$$

If $x_0 < x < x_1$:

$$\int_{-\infty}^{x} \frac{d^n u}{dx^n} (y) \, dy = \int_{-\infty}^{x_0} + \int_{x_0}^{x} = g_0$$

$$\Rightarrow u^{(n)} (x) = u^{(n)} (x_0^+) + g_0, \quad \text{"indep. of } x.$$

cetc.
\[
\frac{d^3u}{dx^3} \text{ is a step function}
\]
\[
\frac{d^2u}{dx^2} \text{ is the integral of a step function}
\]
\[
\text{it is continuous!}
\]

If we knew the slopes at \(x_0, x_1, \ldots, x_n\), we
would have our spline approximation, but
we don't. We can however get the \((n+1)\)
needed slopes \(s_0, s_1, \ldots, s_n\) by using the \((n+1)\)
conditions

\[
\frac{d^2u}{dx^2} \text{ continuous at } x_0, -1, x_n.
\]

On \((x_0, x_1)\), the Hermite cubic is

\[
\begin{align*}
    u(x) &= u_0 \left\{ \frac{2}{h_0^3} (x-x_0)^3 + \frac{3}{h_0} (x-x_0)^2 \right\} \\
         &\quad + s_0 \left\{ \frac{1}{h_0^2} (x-x_0)^3 + \frac{1}{h_0} (x-x_0)^2 \right\} \\
         &\quad + u_1 \left\{ -\frac{3}{h_0^3} (x-x_1)^3 + \frac{3}{h_0} (x-x_1)^2 \right\} \\
         &\quad + s_1 \left\{ \frac{1}{h_0^2} (x-x_1)^3 \right. \left. + \frac{1}{h_0} (x-x_1)^2 \right\}
\end{align*}
\]

\(u_0 = x_0 - x_0\)

\textbf{Exercise} Check that

\[
\begin{align*}
    u(x_0) &= x_0, \\
    u(x_1) &= u_1, \\
    u'(x_0) &= s_0, \\
    u'(x_1) &= s_1
\end{align*}
\]
To the left of \( x_0 \), \( u \) is straight \( \Rightarrow \frac{d^2 u}{dx^2} = 0 \).

At \( x_0 \) we have by continuity \( \frac{d u}{d x} \)
\[
\frac{1}{h_0} (2S_0 + S_1) = \frac{3}{h_0} (u_1 - u_0)
\]

At \( x_1 \)
\[
\frac{1}{h_1} (2S_1 + S_2) + \frac{1}{h_0} (2S_1 + S_0) = \frac{3}{h_1^2} (u_2 - u_1) + \frac{3}{h_0} (u_1 - u_0)
\]

At \( x_k \), \( 1 \leq k \leq n-1 \)
\[
\frac{1}{h_k} (2S_k + S_{k+1}) + \frac{1}{h_{k-1}} (2S_k + S_{k-1}) = \frac{3}{h_k} (u_{k+1} - u_k) + \frac{3}{h_{k-1}} (u_k - u_{k-1})
\]

At \( x_n \)
\[
\frac{1}{h_n} (2S_n + S_{n-1}) = \frac{3}{h_n^2} (u_n - u_{n-1})
\]

All this can be rewritten
\[
2S_k + S_{k+1} \frac{h_k-1}{h_k + h_{k-1}} + 8S_{k-1} \frac{h_k}{h_k + h_{k-1}}
\]

\[
= 3 \frac{h_k-1}{h_k} \frac{1}{h_k + h_{k-1}} (u_{k+1} - u_k) + 3 \frac{h_k}{h_k-1} \frac{1}{h_k + h_{k-1}} (u_k - u_{k-1})
\]

(with the convention that \( S_{-1} \) and \( S_{n+1} = 0 \))
Therefore, we obtain the following system of \((n+1)\) linear equations for the unknowns \(s_0, \ldots, s_n\):

\[
\begin{bmatrix}
2 & a_0 & 0 & \cdots & 0 \\
\mu_1 & a_1 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & a_n \\
\end{bmatrix}
\begin{bmatrix}
s_0 \\
s_1 \\
\vdots \\
s_{n-1} \\
s_n \\
\end{bmatrix} =
\begin{bmatrix}
d_0 \\
d_1 \\
\vdots \\
d_{n-1} \\
d_n \\
\end{bmatrix}
\]

where, for \(k = 1, \ldots, n-1\),

\[
\mu_k = \frac{h_k}{h_k + h_{k-1}} \quad \Lambda_k = \frac{h_{k-1}}{h_k + h_{k-1}}
\]

\[
d_k = 3 \frac{h_{k-1}}{h_k h_k + h_{k-1}} + \frac{1}{h_k h_k + h_{k-1}} - \frac{1}{h_k} \frac{h_{k+1} - h_k}{h_{k+1} h_{k+1} + h_{k+1}}
\]

and \(\mu_0 = 1\), \(d_0 = \frac{3}{h_0} (u_0 - u_0)\),

\[
\mu_n = 1 \quad d_n = \frac{3}{h_{n-1}} (u_n - u_{n-1})
\]
Example

If \( h_k = h \), \( k = 0, 1, \ldots, n \), then

\[
A = \begin{bmatrix}
2 & 1 & 0 \\
\frac{1}{2} & 2 & \frac{1}{2} \\
0 & \frac{1}{2} & 2 \\
\frac{1}{2} & \frac{1}{2} & 1 \\
\end{bmatrix}
\]

We note that \( \lambda_k \geq 0 \) and \( \mu_k \geq 0 \), \( k = 0, -1, \ldots, n \)
and

\[\lambda_k + \mu_k = 1.\]

Theorem

For any partition \( \tau \), \( \tau \neq \tau \), the above system is nonsingular.

Proof.

Let's show that \( 0 \) is not an eigenvalue.

Gershgorin: if \( \lambda \) is an eigenvalue then \( \lambda \in \bigcup_{k} D_k \)

\[D_k = \left\{ z \in \mathbb{C} : |z - a_{kk}| \leq \sum_{j \neq k} |a_{kj}| \right\}\]

Here
The issue of the boundary conditions set to $x_0$ and $x_n$ is important.

Three standard choices

a. $p''(x_0) = 0 \quad p''(x_n) = 0$ \hspace{1cm} \text{simply supported beam.}
   \hspace{1cm} p'(x_0) = 0 \quad p'(x_n) = u_n$
   \hspace{1cm} This is our case.

b. In case the slopes at $x_0$ and $x_n$ are known, then we have
   \hspace{1cm} p'(x_0) = u_0 \quad p'(x_n) = u_n \hspace{1cm} \text{clamped beam.}
   \hspace{1cm} p(x_0) = 0 \quad p(x_n) = 0$

c. Yes another possibility is to consider the "not a hinge" boundary condition (there are implemented in MatLab). It amounts to requiring the continuity of the 3rd derivative at $x_1$, $x_2$, and $x_{n-1}$.

The first and last line of $A$ should be changed accordingly.
Interpolating Lagrange polynomials may not converge to the function whose values they interpolate, as the number of nodes increases.

Interpolating spline functions, however, do converge as the mesh is refined.

Similar results hold for the three b.c. but the simplest form is with boundary condition b.

**Theorem**

If the grid is quasi-uniform, i.e.

\[ \exists \delta > 0, \quad \frac{h_i^2}{\eta_i} \leq \delta, \quad \text{for all } i \]

and if \( u \in C^4([a,b]) \) with \( |u^{(4)}(x)| \leq L \) for any \( x \in [a,b] \), then if \( p \) denotes the unique cubic spline constructed on the above mesh with b.c. of type b, there exists a constant \( C \), independent of \( \eta \), such that

\[ |u(x) - p(x)| \leq C L \eta^4 \]

(4th order method).
Another way to think of splines is to try to construct a "basis" as we did for the Lagrange interpolating polynomials.

First try

\[ x_1 \rightarrow x_2 \rightarrow x_3 \]

Is this possible with a globally $C^2$ piecewise cubic polynomial?

On $(x_1, x_2)$: 4 parameters \[ \Rightarrow 8 \text{ parameters} \]

On $(x_1, x_2)$: 4

At $x_1$: $u_1 = 0$, $s_1 = 0$ + $C^2$ \[ \Rightarrow 3 \text{ conditions} \]

At $x_2$: $u_2 = 1$, $s_2 = 0$ + $C^2$ \[ \Rightarrow 3 \text{ conditions} \]

At $x_3$: $u_3 = 0$, $s_3 = 0$ + $C^2$ \[ \Rightarrow 3 \text{ conditions} \]

Overall.

Second try

To simplify $\rightarrow 0 \ 1 \ 2 \ 3 \ 4$
at $x_0 : u_0 = 0, s_0 = 0 + e^2 \Rightarrow s_1 = 3u_1$

at $x_1 : e^2 \int{\text{...}} \Rightarrow s_1 = 3/4$ (exercise)

at $x_2 : u_2 = 1, s_2 = 0$

By symmetry $s_2 = -3/4, u_3 = 4/4$

B - graph

This type of functions can be used as a basis.
• Every other cubic spline can be expressed as a linear combination of B-splines.

• B-splines can be used as basis functions for other types of methods (for instance FEM: they are both regular (C^2) and relatively local).

• B-splines are widely used in computer graphics applications, medical imaging, computer animation.
Yet another way of approximating functions is to give up completely the idea of being exact at some points, and look instead at the problem of finding $p_n$ such that for $f \in C([a,b])$

$$\int_a^b (f(x) - p_n(x))^2 \, dx$$

is minimal.

This is a continuous version of the least squares method. We are going to generalize this approach by looking at weighted least squares:

For $f \in C([a,b])$, find $p_n$ such that

$$\int_a^b w(x) (f(x) - p_n(x))^2 \, dx$$

is minimal.

An integrable function $w$ is called a weight function on $(a,b)$ if $w(x) > 0$ and $w \neq 0$ on any subinterval of $(a,b)$.

Example: $w(x) = \frac{1}{\sqrt{1-x^2}}$ on $(-1,1)$. 
A weight function can be used to define new norms and new corresponding inner products.

A set of functions \( \{\phi_0, \ldots, \phi_n\} \) is said to be linearly independent on \((a, b)\) if

\[
\sum_{i=0}^{n} c_i \phi_i(x) = 0 \quad \forall x \in (a, b)
\]

\[\Rightarrow \quad c_0 = c_1 = \cdots = c_n = 0.\]

Example

If for each \( j = 0, 1, \ldots, n \), \( \phi_j \) is a polynomial of degree \( j \) (exactly), then \( \{\phi_0, \ldots, \phi_n\} \) is linearly independent on any interval \((a, b)\).

Proof

Let \( p(x) = c_0 \phi_0(x) + c_1 \phi_1(x) + \cdots + c_n \phi_n(x) = 0 \quad \forall x \in (a, b)\)

\[\Rightarrow \quad \text{the coeff. of all the powers of } x \text{ have to be zero, but } c_n \phi_n(x) \text{ is the only term containing } x^n\]

\[\Rightarrow \quad c_n = 0 \Rightarrow c_{n-1} = 0 \Rightarrow \cdots \Rightarrow c_0 = 0.\]
For a given function \( f \in C[\alpha, \beta] \) and for a given set of independent functions \( \phi_0, \ldots, \phi_n \), we want to minimize

\[
\int_{\alpha}^{\beta} \omega(x) (f(x) - p_\alpha(x))^2 \, dx
\]

where \( p_\alpha(x) = \sum_{k=0}^{n} a_k \phi_k(x) \).

Let's consider

\[
E(a_0, a_1, \ldots, a_n) = \int_{\alpha}^{\beta} \omega(x) (f(x) - p(x))^2 \, dx
= \int_{\alpha}^{\beta} \omega(x) (f(x) - \sum_{k=0}^{n} a_k \phi_k(x))^2 \, dx.
\]

The normal equations are

\[
0 = \frac{\partial E}{\partial a_j} = 2 \int_{\alpha}^{\beta} \omega(x)(f(x) - \sum_{k=0}^{n} a_k \phi_k(x))(- \phi_j(x)) \, dx
\]

(\#)

\[
\sum_{k=0}^{n} a_k \int_{\alpha}^{\beta} \omega(x) \phi_k(x) \phi_j(x) \, dx = \int_{\alpha}^{\beta} \omega(x) f_j(x) \, dx, \quad j = 0, \ldots, n.
\]

Verified 2.

Notation: \((u, v) \mapsto \int_{\alpha}^{\beta} \omega(x) u(x) v(x) \, dx\) is an inner product (bilinear, symmetric, positive definite). Consequently

\[
u_2 \mapsto (u, \nu) = \left( \int_{\alpha}^{\beta} \omega(x) u(x) \nu(x) \, dx \right)
\]

is a norm.
We denote these respectively \((\cdot, \cdot)w\) and \(11.011w\).

In order to simplify the normal equations, we are going to try to choose the functions \(\phi_0, -\phi_n\) as orthogonal with respect to the inner product \((\cdot, \cdot)w\).

\[
\int_a^b w(x) \phi_k(x) \phi_j(x) \, dx = \begin{cases} 0 & \text{if } j \neq k \\ x_k > 0 & \text{if } j = k. \end{cases}
\]

The normal equations \((\star)\) then become

\[
a_j a_j = \int_a^b w(x) f(x) \phi_j(x) \, dx
\]

\[
\Rightarrow a_j = \frac{1}{x_j} \int_a^b w(x) f(x) \phi_j(x) \, dx, \quad j = 0, -1, \ldots.
\]

Therefore, we have proved

Theorem

If \(\phi_0, -\phi_n\) is an orthogonal set of functions on \([a, b]\) with respect to a weight function \(w\), then the least squares approximation to \(f\) on \([a, b]\) with respect to \(w\) is

\[
P(x) = \sum_{k=0}^\infty a_k \phi_k(x)
\]

with \(a_k = \frac{\int_a^b w(x) f(x) \phi_k(x) \, dx}{\int_a^b w(x) \phi_k^2(x) \, dx} = \frac{(f, \phi_k)w}{(\phi_k, \phi_k)w}\).
Example 1 (very important)

\[ \phi_k(x) = e^{ikx} (= \cos kx + i \sin kx) \]

Notation: \[ (f,g) = \int_0^{2\pi} f(x) \overline{g(x)} \, dx \]

\[ \|f\| = (f \cdot f)^{1/2} \]

1. The above functions are orthogonal, with respect to the above inner product \((W = 1)\)

\[ (\phi_k, \phi_l) = \int_0^{2\pi} e^{ikx} e^{-ilx} \, dx = \frac{e^{i(k-l)x}}{i(k-l)} \bigg|_0^{2\pi} = 0, \quad k \neq l \]

\[ \|\phi_k\|^2 = \int_0^{2\pi} e^{ikx} e^{-ikx} \, dx = 2\pi. \]

2. They are complete; no function can be orthogonal to all of them (without being zero).

For a given \(f\) (say, absolutely continuous), we can consider

\[ f = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad \text{Fourier series} \]

where, as before

\[ c_k = \frac{(\phi_k, f)}{(\phi_k, \phi_k)} = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} \, dx. \]
(In the chapter on Fourier, we will ask and answer the questions:
- when and in what sense is the series convergent?
- what is the relationship between the series and \( f \)?
- how rapidly does the series converge?)

3. They are convenient; coefficients are easy to compute, and every harmonic \( e^{ikx} \) is an eigenfunction of every derivative and every finite difference:

\[
\frac{d}{dx} e^{ikx} = ike^{ikx}
\]

The Laplacian has eigenvalues:

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) e^{ik(x+ky)} = -\left( k_x^2 + k_y^2 \right) e^{ik(x+ky)}
\]

One-sided and centered finite differences have eigenvalues:

\[
\begin{align*}
&\frac{e^{ik(x+h)} - e^{ikx}}{h} = \frac{e^{ikx} - e^{ikh}}{h} = \frac{i \sin kh}{h} e^{ikh} \\
&\frac{e^{ikh} - e^{ik(x-h)}}{2h} = \frac{i \sinh kh}{h} e^{ikh}
\end{align*}
\]
Example \[ f(x) = \frac{3}{5 - 4\cos x} \]

\( f \) is infinitely differentiable & periodic with all its derivatives on \([0, 2\pi]\).

\[ e_k = \frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} = \frac{1}{2\pi} \int_0^{2\pi} \frac{3}{5 - 4\cos x} e^{-ikx} \, dx \]

\[ = \frac{3}{2\pi} \int_0^{2\pi} \cos kx \, dx = \frac{3\pi}{2\pi} \int_0^{2\pi} \sin kx \, dx \]

\[ = \frac{3\pi}{2\pi} \int_0^{2\pi} \cos kx \, dx = 0 \]

\[ = \frac{3\pi}{2\pi} \int_0^{2\pi} \sin kx \, dx \]

\[ = \frac{3\pi}{2\pi} \int_0^{2\pi} \frac{\sin kx}{5 - 4\cos x} \, dx \]

\[ = \left( \frac{3\pi}{4} \frac{\pi}{a^2 - 1} \right) k = -1 \] (Gradshteyn & Ryzhik, p. 366, (a=2))

Therefore! The coeff. \( c_k \) decrease exponentially (faster than any negative power of \( k \)). If \( f \) is not that smooth, \( c_k \) does not diminish this fast! (see Fig 6)

\((\text{if } f \text{ real } \Rightarrow c_{-k} = \overline{c_k} \Rightarrow \text{Fourier})\)
Example 2: systems of orthogonal polynomials.

First, we note that \( \{1, x, \ldots, x^n\} \) is a basis for \( P_n \).

On a given interval \((a, b)\), we want to construct an orthogonal basis, i.e.

\[
(f_k, f_j)_w = \int_a^b w(x) f_k(x) f_j(x) \, dx = \int_a^b w(x) f_k(x) f_j(x) \, dx = \delta_{k,j} \cdot \beta_k.
\]

We use the Gram-Schmidt process in order to transform \( \{1, x, x^2, \ldots\} \) into an orthogonal set of functions.

Let us call pre-factor \( \phi_0(x) \equiv 1 \) on \([a, b]\).

\[
\phi_1(x) = x - \frac{(\phi_0, x)_w}{\|\phi_0\|_w^2} \phi_0(x).
\]

It is easy to check:

\[
(\phi_1, \phi_1)_w = (x, \phi_1)_w - \frac{(\phi_0, x)_w (\phi_0, \phi_1)_w}{\|\phi_0\|_w^2} = 0
\]

Similarly:

\[
\phi_k(x) = x^k - \frac{(\phi_0, x^k)_w}{\|\phi_0\|_w^2} \phi_0(x) - \frac{(\phi_1, x^k)_w (\phi_1, \phi_1)_w}{\|\phi_1\|_w^2} \phi_1(x) - \ldots - \frac{(\phi_{k-1}, x^k)_w (\phi_{k-1}, \phi_{k-1})_w}{\|\phi_{k-1}\|_w^2} \phi_{k-1}(x)
\]
Each different choice of \( w \) and \([a, b]\) lead to a new family of polynomials.

\( [a, b] = [-1, 1] \quad w(x) = (1-x)^\alpha (1+x)^\beta \quad \alpha, \beta > -1 \)

**Jacobi polynomials**

\( \alpha = \beta = 0, \quad (w = 1) : \quad \text{Legendre polynomials} \)

\( \alpha = \beta = -\frac{1}{2}, \quad (w(x) = (1-x^2)^{-1/2}) : \quad \text{Chebyshev polynomials} \)

**Example:** Legendre pol.

\[
\phi_0(x) = 1
\]

\[
\phi_1(x) = x - \frac{\langle \phi_0, x \rangle w}{w \phi_0(x)}
\]

\[
\phi_2(x) = x^2 - \frac{\langle \phi_0, x^2 \rangle w}{w \phi_0(x)} \phi_0(x) - \frac{\langle \phi_1, x^2 \rangle w}{w \phi_1(x)} \phi_1(x)
\]

\[
\phi_3(x) = x^3 - \frac{\langle \phi_0, x^3 \rangle w}{w \phi_0(x)} \phi_0(x) - \frac{\langle \phi_1, x^3 \rangle w}{w \phi_1(x)} \phi_1(x) - \frac{\langle \phi_2, x^3 \rangle w}{w \phi_2(x)} \phi_2(x)
\]
\begin{align*}
(\phi_0, x^2) &= \int_{-1}^{1} x^2 \, dx = \frac{2}{3}, \quad \|\phi_0\|^2 = 2 \\
(\phi_1, x^2) &= \int_{-1}^{1} x^3 \, dx = 0, \quad \|\phi_1\|^2 = \int_{-1}^{1} x^2 \, dx = \frac{2}{3} \\
\Rightarrow \quad \phi_2(x) &= x^2 - \frac{1}{3}
\end{align*}

Systems of orthogonal polynomials can be used in order to

1. approximate functions, see assignment #2,
2. design methods for solving PDE's
3. design methods for numerical integration.