**Definition:** In each of the problems we use the following:

1. \( f(E) = \{ f(x) \mid x \in E \} \) and
2. \( f^{-1}(F) = \{ x \mid f(x) \in F \} \)

where \( E \subseteq X \) and \( F \subseteq Y \).

1. Suppose \( f : X \to Y \) and that \( A \subseteq X \).
   To show that \( A \subseteq f^{-1}(f(A)) \), assume that \( x \in A \). By 1 in the above definition, this means that \( f(x) \in f(A) \). However by 2 above, this means that \( x \in f^{-1}(f(A)) \). This concludes the inclusion portion of the problem.

   Now suppose that the two are equal for all \( A \subseteq X \), we will show that \( f \) must be injective. Suppose otherwise and that \( a, b \in X \), with \( a \neq b \), and \( f(a) = f(b) \). Let \( A = \{ a \} \). In this case \( f(A) = \{ f(a) \} \), and \( f^{-1}(f(A)) = \{ a, b \} \neq A \). This contradiction implies that \( f \) must be one-to-one.

   So suppose that \( f \) is one-to-one. We will show that the two sets are equal. We need only show that \( f^{-1}(f(A)) \subseteq A \). So let \( x \in f^{-1}(f(A)) \). Again using part 2 of the above definition, \( x \in f^{-1}(f(A)) \) implies that \( f(x) \in f(A) \). If this is true \( f(x) = f(y) \) for some \( y \in A \). Since \( f \) is one-to-one, \( x = y \in A \), and we are done.

2. For this problem we must start by proving \( f(f^{-1}(B)) \subseteq B \) if \( B \subseteq Y \). To this end let \( y \in f(f^{-1}(B)) \). This means that there is an \( x \in f^{-1}(B) \) so that \( f(x) = y \). But by the definition part 2, \( y = f(x) \in B \).

   As before it is easy to show that if \( f(f^{-1}(B)) = B \), for all \( B \subseteq Y \), then \( f \) must be onto. Suppose \( f \) is not onto. Then there is a \( y \in Y \) which is not the image of any \( x \) in \( X \). For this \( y \) let \( B = \{ y \} \). Since nothing maps onto \( y \), \( f^{-1}(B) = \emptyset \). Thus \( f(f^{-1}(B)) = \emptyset \neq \{ y \} = B \).

   So suppose that \( f \) is onto (surjective). Let \( y \in B \). Since \( f \) is onto, there is an \( x \in X \) so that \( f(x) = y \). In other words, an \( x \) so that \( x \in f^{-1}(\{ y \}) \subseteq f^{-1}(B) \). This means that \( y = f(x) \in f(f^{-1}(B)) \). Hence \( B \subseteq f(f^{-1}(B)) \).

The next two exercises are very similar to these two, except perhaps easier.