Parametric Programming and Post-Optimality Analysis

Formulas that express the optimal value and an optimal solution as functions of the input data (i.e., the problem parameters) are extremely important:

(1) for determining the accuracy needed in the input data to achieve the desired accuracy in the output data (i.e., the desired accuracy for the optimal value and an optimal solution),

(2) for determining the effect that changing external conditions would have on the optimal value and an optimal solution.

To derive such formulas, assume that enough slack variables have been introduced to place the problem in standard form. The resulting standard formulation obviously uses only the original input data — namely, particular values for the problem parameters $A, b, c, d$ appearing in the resulting "input tableau"

$$M_0 \triangleq \begin{bmatrix} d & c \\ b & A \end{bmatrix}$$

Needless to say, the preceding input tableau $M_0$ is generally not canonical; and even if it is, it need not be feasible. In any event, we continue to assume that all variables, other than the objective variable, are restricted to be non-negative. Moreover, we note that elementary row operations can always be used to transform $M_0$ into an initial canonical tableau $M_1$ for either a phase-1 calculation or (if one is more fortunate) a phase-2 calculation. Of course, the required elementary row operations, including those in any subsequent phase-1 and/or phase-2 calculations, utilize the original input data for $A, b, c, d$. 
If the original input data for $A, b, c, d$ describes a problem that is consistent and bounded, we know the simplex method (with a preventive for circling) eventually terminates in phase 2 with step 1, producing an optimal tableau $M_t$ with a terminal basic sequence $T$.

Post-optimality analysis treats only this situation, and then only in the case that $M_t$ and $M_0$ are the same size. Actually, a detailed review of the simplex method in its entirety should convince the reader that $M_t$ cannot be larger than $M_0$ and is, in fact, smaller than $M_0$ only when there are linearly dependent rows in $M_0$ (a situation that would, of course, be detected when $M_0$ is transformed into $M_t$ via elementary row operations). In retrospect, such rows can be deleted, producing a deleted $M_0$ of the same size as $M_t$. However, any postoptimality analysis performed with such an $M_0$ and $M_t$ implicitly assumes that the row deletions remain valid during the desired variations in $A, b, c, d$ from their original input values -- an assumption that has to be validated in any particular situation before the ensuing post-optimality analysis can be completely justified.

In summary, without much loss of generality, we assume that the input tableau $M_0$ with the original input data for $A, b, c, d$ is transformed, via the simplex method, into an optimal tableau $M_t$ with a terminal basic sequence $T$ and with the same size as $M_0$ (where $M_0$ itself need not have a basic sequence $\sigma$ because it need not be canonical). Then, $M_0$ and $M_t$ are equivalent matrices, which means there is a nonsingular matrix $Q_{0t}$ such that

$$M_0 = Q_{0t} M_t \quad \text{and} \quad M_t = Q_{0t}^{-1} M_0.$$
Recalling that \( M_0 \) and \( M_t \) are actually submatrices of the equivalent tableau matrices \([M_0 \ u]\) and \([M_t \ u]\) containing the invariant unit column vector \( u \triangleq (1, 0, \ldots, 0)^T \), we realize that the equation \( M_0 = Q_{0t} M_t \) is only part of the more complete equation

\[
[M_0 \ u] = Q_{0t} [M_t \ u],
\]

from which we deduce that column 0 of \( Q_{0t} \) is simply \( u \). Similarly, recalling the unit vector nature of the basic columns of \( M_t \), we further infer from the equation \( M_0 = Q_{0t} M_t \) that

\[
Q_{0t} = [u \ M_0^T] = \begin{bmatrix} 1 & c^T \\ 0 & A^T \end{bmatrix} \quad \text{and hence} \quad Q_{0t}^{-1} = \begin{bmatrix} 1 & c^T^{-1} \\ 0 & A^{-1} \end{bmatrix}
\]

Now, the preceding formula for \( Q_{0t}^{-1} \) implies, by virtue of the theory of determinants, that the submatrix \( A^T \) has an inverse \( (A^T)^{-1} \); with which it is easy to verify that

\[
\begin{bmatrix} 1 & c^T \\ 0 & A^T \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -c^T (A^T)^{-1} \\ 0 & (A^T)^{-1} \end{bmatrix}
\]

Finally, multiplying the preceding formula for \( Q_{0t}^{-1} \) into the defining formula for \( M_0 \) gives, via the equation \( M_t = Q_{0t}^{-1} M_0 \), the formula
\[ M_t = \begin{bmatrix} \frac{d - c^T [A^T]^{-1} b}{[A^T]^{-1} b} & \frac{c - c^T [A^T]^{-1} A}{[A^T]^{-1} A} \end{bmatrix} \]

From the preceding formula for \( M_t \) and from the assumed optimality of \( M_t \) (including the implied feasibility of \( M_t \)), we conclude that with the original input data for \( A, b, c, d \):

1. \( A^T \) has an inverse \([A^T]^{-1}\),
2. \([A^T]^{-1} b\) has components that are non-negative,
3. \( c - c^T [A^T]^{-1} A \) has nonbasic components that are non-positive when maximizing (non-negative when minimizing).

Moreover, since matrix algebra implies that

\[ [c - c^T [A^T]^{-1} A]^T = [c^T - c^T [A^T]^{-1} [A^T]] = [c^T - c^T I] = [c^T - c^T] = 0 \]

and that

\[ [[A^T]^{-1} A]^T = [A^T]^{-1} A^T = I \]

for all values of \( A, b, c, d \) for which \( A^T \) is invertible, we infer from the resulting "matrix equivalence" of the displayed formulas for \( M_0 \) and \( M_t \) that the preceding conditions (1), (2) and (3) are not only necessary but are also sufficient for \( M_t \) to be optimal (and hence terminal).
In working with condition (3), it is helpful to let $T'$ denote the "nonbasic sequence" complementing the basic sequence $T$ (namely, the indices not in $T$, arranged in "natural order")

Then, matrix algebra implies that the nonbasic components of $c - c^T [A^T]^{-1} A$

are simply the components of $[c - c^T [A^T]^{-1} A]^{T'} = [c^{T'} - c^T [A^T]^{-1} A^{T'}].$

In summary, the problem with input tableau

$$H_0 \equiv \begin{bmatrix} d & c \\ b & A \end{bmatrix}$$

also has an optimal tableau

$$H_T \equiv \begin{bmatrix} d - c^T [A^T]^{-1} b & c - c^T [A^T]^{-1} A \\ [A^T]^{-1} b & [A^T]^{-1} A \end{bmatrix}$$

with basic sequence $T$

if, and only if,

(1) $A^T$ has an inverse $[A^T]^{-1},$

(2) $[A^T]^{-1} b \geq 0, $

(3) $[c^{T'} - c^T [A^T]^{-1} A^{T'}] \leq 0 (< 0),$

conditions that are satisfied by the original input data and infinitely many other sets of input data.
Consequently, on the domain

\[((A, b, c, d) \mid A^T \text{ has an inverse } [A^T]^{-1}; [A^T]^{-1} b \geq 0; \text{ and }\]

\[c^T - c^T [A^T]^{-1} A^T' \leq 0 (\geq 0)\]

the formulas that express the optimal value \(f^*\) and an optimal solution \(x^*\) as functions of the parameters \(A, b, c,\) and \(d\) appearing in the input tableau are

\[f^* = -d + c^T [A^T]^{-1} b,\]

\[x^*^T = [A^T]^{-1} b,\]

\[x^*^T' = 0.\]

The lack of any restrictions on \(d\), and the particular dependence of \(f^*\) and \(x^*\) on \(d\)(actually, a lack of dependence in the case of \(x^*\)) are as expected. However, the lack of any dependence of \(f^*\) and \(x^*\) on \(A^T'\) and \(c^T'\) (except those dependencies determined by their domain) may not have been expected. Note also that \(x^*^T'\) is identically zero (as expected) and that the formula for \(x^*\) involves only \(A^T\) and \(b\). Furthermore, for a fixed \(A\) for which \(A^T\) has an inverse, \(x^*\) is just a linear transformation of \(b\) and \(f^*\) is just \(-d\) plus a bilinear function of \(b\) and \(c^T\), both on the restricted domain

\[\{b \mid [A^T]^{-1} b \geq 0\} \times \{c \mid [c^T' - c^T [A^T]^{-1} A^T'] \leq 0 (\geq 0)\}.\]
The preceding formulas show that \((A^T)^{-1} A^T\) and \((A^T)^{-1} A^T'\) are the only nontrivial matrices needed to describe the dependence of \(f^*\) and \(x^*\) on \((b,c)\) as \((b,c)\) varies over its (restricted) domain. Computationally, it is important to realize that these two matrices need not always be obtained via a numerical inversion of \(A^T\) followed by a multiplication of \((A^T)^{-1}\) into \(A^T'\). In particular, assuming that

\[ A_c \] denotes the constraint coefficient submatrix of the terminally optimal tableau \(M_c\),

first note from the previously derived representation formula for \(M_c\) that

\[ A_c = [A^T]^{-1} A. \]

Then, note from this formula that

\[ A_c^T' = [(A^T)^{-1} A]^T' = [A^T]^{-1} A^T' \]

so

\[ [A^T]^{-1} A^T' = A_c^T' \],

which shows that the columns of \((A^T)^{-1} A^T'\) are just appropriately selected columns of the terminal constraint coefficient submatrix \(A_c\).

Similarly, if the input tableau \(M_0\) is canonical with a basic sequence \(\sigma\), note from the formula \(A_c = [A^T]^{-1} A\) that

\[ A_c^\sigma = [(A^T)^{-1} A]^\sigma = [A^T]^{-1} A^\sigma = [A^T]^{-1} I = [A^T]^{-1} \]

so

\[ [A^T]^{-1} = A_c^\sigma \]

if \(M_0\) is canonical with a basic sequence \(\sigma\).

If \(M_0\) is not canonical, the columns of \((A^T)^{-1}\) need not be appropriately selected columns of the terminal constraint coefficient submatrix \(A_c\); in which case \((A^T)^{-1}\) must be obtained via a numerical inversion of \(A^T\) (even though \((A^T)^{-1} A^T' = A_c^T'\)).
Of course, the preceding formulas must be constructed for each possible choice of \( T \) in order to obtain a complete description of the dependence of \( f^* \) and \( x^* \) on \( A, b, c, d \). Needless to say, there are only a finite number (though generally a very large number) of such possible choices of \( T \); and those input vectors \( (A, b, c, d) \) that are associated with a consistent bounded problem will lie in at least one of the corresponding domains. Naturally, the precise nature of a given input vector \( (A, b, c, d) \) can always be determined via the simplex method.

In fact, post-optimality analysis is simply the use of an optimal tableau (actually, just the corresponding terminal basic sequence \( T \)) to determine a domain in which the given input vector \( (A, b, c, d) \) lies, so that the corresponding formulas for \( f^* \) and \( x^* \) can then be used to analyze the effect on optimality of "small changes" in the given input vector \( (A, b, c, d) \). Since small changes tend to occur only in \( b, c, \) and \( d \) with \( A \) remaining fixed (e.g., the small changes that occur in resource availability and product prices, with the technology remaining fixed), most post-optimality analyses are concerned only with the changes in \( f^* \) and \( x^* \) induced by small changes in \( b, c, \) and \( d \), with \( A \) remaining fixed.

Needless to say, such changes can be derived via the preceding formulas and the differential calculus. In particular,

\[
(\text{I}) \quad \text{if } [A^T]^{-1} b > 0, \text{ then } \nabla_b f^* = c^T [A^T]^{-1} b
\]

\[
(\text{II}) \quad \text{if } [c^T + c^T A^T [A^T]^{-1} a^T] < 0 (> 0),
\]

\[
\text{then } \nabla_{c^T} f^* = [A^T]^{-1} b \quad \text{and} \quad \nabla_{c^T} f^* = 0
\]

\[
(\text{III}) \quad \nabla_d f^* = -l
\]
The preceding statements (I) and (II) can be greatly simplified with the aid of the following observations:

\[ (A^T)^{-1}b = b_c, \text{ where } b_c \text{ denotes the distinguished column of } \]

the terminally optimal tableau \( M_t \).

\[
[c'' - c^T (A^T)^{-1} a_t'] = [c - c^T (A^T)^{-1} a] a_t' = c c_t', \]

where

\( c_c \) denotes the distinguished row of \( M_t \).

Moreover, when the input tableau \( M_0 \) is canonical with a basic sequence \( \sigma \), statement (I) can be further simplified with the aid of the following observation:

\[
c^T (A^T)^{-1} = -[0 - c^T (A^T)^{-1} I]
= -[c^\sigma - c^T (A^T)^{-1} A^\sigma]
= -[c - c^T (A^T)^{-1} A] \sigma
= -c c_t'.
\]

In summary, the simplified statements are:

(I') if \( b_c > 0 \), then \( c_c f* = c^T (A^T)^{-1} \), which is just
\(-c c_t' \) when the input tableau \( M_0 \) is canonical with a
basic sequence \( \sigma \).

(II') if \( c_t' f* < 0 \) (\( > 0 \)), then \( c c_t' f* = b_c \) and
\( c c_t' f* = 0 \).

(III') \( v_d f* = -1 \).
If one of the preceding inequalities is not satisfied (strictly), the corresponding gradient may not exist (even though the corresponding nonstrict inequality is satisfied). However, since \( b^* \neq 0 \) only when the optimal solution \( x^* \) is degenerate, we infer that \( \nabla_b f^* \) exists and is given by the formula in statement \((I')\) when \( x^* \) is not degenerate (which is usually the situation). Moreover, "duality theory" will eventually enable us to make an analogous assertion about statement \((II')\).

Finally, for those \((b,c)\) for which \( [A^T]^{-1} b \geq 0 \) but \( [c - \sigma [A^T]^{-1} A] ^T \not\leq 0 \) \((\not\geq 0)\), it should be noted that one can simply carry out enough additional pivots on \( M_e \) to obtain a new optimal tableau \( M_e \) and basic sequence \( T \) before applying the preceding post-optimality analysis. Moreover, "duality theory" will eventually enable us to make an analogous observation about those \((b,c)\) for which \( [c - \sigma [A^T]^{-1} A] ^T \not\leq 0 \) \((\not\geq 0)\) but \( [A^T]^{-1} b \not\geq 0 \).

In the following examples of post optimality analysis, we assume that the given \( A \) remains fixed while \( b, c \) and \( \sigma \) vary from their given values.
Example
(The Resource Allocation Problem)

\[
M_0 = \begin{bmatrix}
0 & 40 & 80 & 0 & 0 & 0 \\
180 & 2 & 4/5 & 1 & 0 & 0 \\
120 & 1 & 1 & 0 & 1 & 0 \\
1000 & 0 & 10 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[\sigma = (3, 4, 5)\]

\[
M_t = \begin{bmatrix}
-8800 & 0 & 0 & 0 & -40 & -4 \\
60 & 0 & 0 & 1 & -2 & 3/25 \\
20 & 1 & 0 & 0 & 1 & -1/10 \\
100 & 0 & 1 & 0 & 0 & 1/10 \\
\end{bmatrix}
\]

\[T = (3, 1, 2)\]

Now,

\[
[A^T]^{-1} = \begin{bmatrix}
0 & 0 & 1 & -2 & 3/25 \\
1 & 0 & 0 & 1 & -1/10 \\
0 & 1 & 0 & 0 & 1/10 \\
\end{bmatrix}
\]

\[
\sigma = \begin{bmatrix}
1 & -2 & 3/25 \\
0 & 1 & -1/10 \\
0 & 0 & 1/10 \\
\end{bmatrix}
\]

and

\[
[A^T]^{-1} A^{T'} = \begin{bmatrix}
0 & 0 & 1 & -2 & 3/25 \\
1 & 0 & 0 & 1 & -1/10 \\
0 & 1 & 0 & 0 & 1/10 \\
\end{bmatrix}
\]

\[
T' = \begin{bmatrix}
-2 & 3/25 \\
1 & -1/10 \\
0 & 1/10 \\
\end{bmatrix}
\]
Consequently,

\[ f^* = -\dot{b} + [c_3 \ c_1 \ c_2] \begin{bmatrix} 1 & -2 & 3/25 \\ 0 & 1 & -1/10 \\ 0 & 0 & 1/10 \end{bmatrix} \begin{bmatrix} \dot{b}_1 \\ \dot{b}_2 \\ \dot{b}_3 \end{bmatrix} \]

\[ = -\dot{b} + [c_3 \ c_1 \ c_2] \begin{bmatrix} \dot{b}_1 - 2 \dot{b}_2 + (3/25) \dot{b}_3 \\ \dot{b}_2 - (1/10) \dot{b}_3 \\ (1/10) \dot{b}_3 \end{bmatrix} \]

\[ = -\dot{b} + c_3 [b_1 - 2 \dot{b}_2 + (3/25) \dot{b}_3] + c_1 [\dot{b}_2 - (1/10) \dot{b}_3] + c_2 [(1/10) \dot{b}_3] \]

and

\[
\begin{bmatrix} x_3^* \\ x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3/25 \\ 0 & 1 & -1/10 \\ 0 & 0 & 1/10 \end{bmatrix} \begin{bmatrix} \dot{b}_1 \\ \dot{b}_2 \\ \dot{b}_3 \end{bmatrix} = \begin{bmatrix} \dot{b}_1 - 2 \dot{b}_2 + (3/25) \dot{b}_3 \\ \dot{b}_2 - (1/10) \dot{b}_3 \\ (1/10) \dot{b}_3 \end{bmatrix} \begin{bmatrix} x_4^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} x_4^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ,
\]

provided that

\[
\begin{bmatrix} b_1 - 2 b_2 + (3/25) b_3 \\ b_2 - (1/10) b_3 \\ (1/10) b_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

and

\[
\begin{bmatrix} b_1 - 2 b_2 + (3/25) b_3 \\ b_2 - (1/10) b_3 \\ (1/10) b_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]
\[ [c_4, c_5] - [c_3, c_1, c_2] \begin{bmatrix} -2 & 3/25 \\ 1 & -1/10 \\ 0 & 1/10 \end{bmatrix} = [c_4, c_5] - [-2c_3 + c_1, (3/25)c_3] \]

\[ - (1/10)c_1 + (1/10)c_2] = [c_4 + 2c_3 - c_1, c_5 - (3/25)c_3] \]

\[ + (1/10)c_1 - (1/10)c_2] \leq [0, 0] \]

Since

\[ b_t = \begin{bmatrix} 60 \\ 20 \\ 100 \end{bmatrix} > 0 \] and \[ c_t^T = [40, 4] < [0, 0], \]

post-optionality analysis asserts that

\[ \nabla_b f^* = -c_t \sigma = [0, 40, 4] = [f^*_1, f^*_2, f^*_3] \]

\[ \nabla_{c_t} f^* = b_t = \begin{bmatrix} 60 \\ 20 \\ 100 \end{bmatrix} = [f^*_1, f^*_2, f^*_3] \] and \[ \nabla_{c_t^T} f^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = [f^*_4, f^*_5] \]

and

\[ \nabla_d f^* = -1 = f^*_d. \]
Example
(The Diet Problem)

\[ M_0 = \begin{bmatrix}
0 & 2 & 2 & 10 & 0 & 0 & 0 & 0 \\
13 & 2 & 4 & 2 & -1 & 0 & 0 & 0 \\
10 & 2 & 1 & 3 & 0 & -1 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
5/2 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \]

\[ M_c = \begin{bmatrix}
-77/3 & 0 & 0 & 0 & 1/3 & 2/3 & 22/3 & 0 \\
7/6 & 1 & 0 & 0 & 1/6 & -2/3 & 5/3 & 0 \\
5/3 & 0 & 1 & 0 & -1/3 & 1/3 & -1/3 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
4/3 & 0 & 0 & 0 & -1/6 & 2/3 & -5/3 & 1
\end{bmatrix} \]

\( T = (1, 2, 3, 7) \)
Now,

\[
[A^T]^{-1} = \begin{bmatrix}
2 & 4 & 2 & 0 \\
2 & 1 & 3 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
\end{bmatrix}^{-1} = \begin{bmatrix}
-1/6 & 2/3 & -5/3 & 0 \\
1/3 & -1/3 & 1/3 & 0 \\
0 & 0 & 1 & 0 \\
1/6 & -2/3 & 5/3 & 1 \\
\end{bmatrix}
\]

and

\[
[A^T]^{-1} A^T = \begin{bmatrix}
1 & 0 & 0 & 1/6 & -2/3 & 5/3 & 0 \\
0 & 1 & 0 & -1/3 & 1/3 & -1/3 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1/6 & 2/3 & -5/3 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1/6 & -2/3 & 5/3 \\
-1/3 & 1/3 & -1/3 \\
0 & 0 & -1 \\
-1/6 & 2/3 & -5/3 \\
\end{bmatrix}
\]

Consequently,

\[
f^* = -d + [c_1 c_2 c_3 c_4] \begin{bmatrix}
-1/6 & 2/3 & -5/3 & 0 \\
1/3 & -1/3 & 1/3 & 0 \\
0 & 0 & 1 & 0 \\
1/6 & -2/3 & 5/3 & 1 \\
\end{bmatrix} \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
\end{bmatrix}
\]

\[
= -d + [c_1 c_2 c_3 c_4] \begin{bmatrix}
-(1/6)b_1 + (2/3)b_2 - (5/3)b_3 \\
(1/3)b_1 - (1/3)b_2 + (1/3)b_3 \\
(1/6)b_1 - (2/3)b_2 + (5/3)b_3 + b_4 \\
\end{bmatrix}
\]
\[\begin{align*}
&= -d + c_1[-(1/6)b_1 + (2/3)b_2 - (5/3)b_3] + c_2[(1/3)b_1 - (1/3)b_2 + (1/3)b_3] \\
&\quad + c_3 b_3 + c_7[(1/6)b_1 - (2/3)b_2 + (5/3)b_3 + b_4]
\end{align*}\]

and

\[
\begin{bmatrix}
\times_1^* \\
\times_2^* \\
\times_3^* \\
\times_4^*
\end{bmatrix} = \begin{bmatrix}
-1/6 & 2/3 & -5/3 & 0 \\
1/3 & -1/3 & 1/3 & 0 \\
0 & 0 & 1 & 0 \\
1/6 & -2/3 & 5/3 & 1
\end{bmatrix} \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix} = \begin{bmatrix}
-(1/6)b_1 + (2/3)b_2 - (5/3)b_3 \\
(1/3)b_1 - (1/3)b_2 + (1/3)b_3 \\
(1/6)b_1 - (2/3)b_2 + (5/3)b_3 + b_4
\end{bmatrix}
\]

while

\[
\begin{bmatrix}
\times_4^* \\
\times_5^* \\
\times_6^*
\end{bmatrix} = \times_2^{* T'} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

provided that

\[
\begin{bmatrix}
-(1/6)b_1 + (2/3)b_2 - (5/3)b_3 \\
(1/3)b_1 - (1/3)b_2 + (1/3)b_3 \\
(1/6)b_1 - (2/3)b_2 + (5/3)b_3 + b_4
\end{bmatrix} \geq \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
c_4 & c_5 & c_6 \\
1/6 & -2/3 & 5/3 \\
-1/3 & 1/3 & -1/3 \\
0 & 0 & -1 \\
-1/6 & 2/3 & -5/3
\end{bmatrix}
\]
\[
= \begin{bmatrix} c_4 & c_5 & c_6 \end{bmatrix} - \begin{bmatrix} (1/6)c_1 - (1/3)c_2 - (1/6)c_7 - (2/3)c_4 + (1/3)c_2 \\
(2/3)c_7 - (5/3)c_1 - (1/3)c_2 - c_3 - (5/3)c_7 \end{bmatrix} = \begin{bmatrix} c_4 - (1/6)c_1 \\
(1/3)c_2 + (1/6)c_7 - c_3 + (2/3)c_7 - (1/3)c_2 - (1/3)c_2 - (5/3)c_7 \\
(1/3)c_2 + c_3 + (5/3)c_7 \end{bmatrix} \geq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.
\]

Since
\[
b_t = \begin{bmatrix} 7/6 \\
5/3 \\
2 \\
4/3 \\
0
\end{bmatrix} > \begin{bmatrix} 0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
\[c_t^{T'} = \begin{bmatrix} 1/3 & 2/3 & 22/3 \end{bmatrix} > \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},
\]

post-optimality analysis asserts that
\[ \begin{bmatrix}
-1/6 & 2/3 & -5/3 & 0 \\
1/3 & -1/3 & 1/3 & 0 \\
0 & 0 & 1 & 0 \\
1/6 & -2/3 & 5/3 & 1
\end{bmatrix} \]

\[ \mathbf{v}_b \mathbf{f}^* = [2 \ 2 \ 10 \ 0] \begin{bmatrix}
1/3 & -1/3 & 1/3 & 0 \\
0 & 0 & 1 & 0 \\
1/6 & -2/3 & 5/3 & 1
\end{bmatrix} = [1/3 \ 2/3 \ 22/3 \ 0] \]

\[ = [f_{b1}^* \ f_{b2}^* \ f_{b3}^* \ f_{b4}^*] \]

\[ \mathbf{v}_{cT}^T \mathbf{f}^* = \mathbf{b}_t = \begin{bmatrix}
7/6 \\
5/3 \\
2 \\
4/3
\end{bmatrix} = \begin{bmatrix}
\mathbf{f}_{c1}^* \\
\mathbf{f}_{c2}^* \\
\mathbf{f}_{c3}^* \\
\mathbf{f}_{c7}^*
\end{bmatrix} \quad \text{and} \quad \mathbf{v}_{cT}^T \mathbf{f}^* = \mathbf{0} = \begin{bmatrix}
\mathbf{f}_{c4}^* \\
\mathbf{f}_{c5}^* \\
\mathbf{f}_{c6}^*
\end{bmatrix} \]

and

\[ \mathbf{v}_d \mathbf{f}^* = -1 = \mathbf{f}_d^*. \]

**Exercise**: Do a post-optimality analysis for the scaffold-problem example (using the input and terminal tableaus corresponding to the input and terminal schemas you presumably generated during your solution via the simplex method in a previous exercise).