Mixing and Flow Redistribution
(Subchannel Analysis)

Flow channels formed by adjacent rods in rod bundles are open to each other through the gaps between the rods, allowing the flow in one channel to mix with that of others.

Pressure differences between the channels result in a flow redistribution allowing for the transfer of mass, energy and momentum between the subchannels. Models used to predict the thermal hydraulic conditions in rod bundles must then account for these cross flows. Two types of cross flow will be considered in our development of the subchannel equations: (1) Pressure driven convective flows due to the pressure difference between the channels which result in a net mass transfer between the subchannels, and (2) Turbulent or eddy mixing flows which do not result in a net mass transfer but do transfer energy and momentum.

Derivation of the Subchannel Equations

We begin by considering a control volume of length $\Delta z$ and area $A_k$ where $A_k$ is the cross sectional flow area of the channel.

Figure 1: Subchannel in a rectangular lattice

Figure 2: Control Volume for Subchannel Analysis
Mass Balance

Consider first a mass balance on the control volume $V_k = A_k \Delta z$

$$\frac{dM_k}{dt} = \sum_i \dot{m}_i - \sum_e \dot{m}_e$$  (1)

Define:  
\begin{align*}
\dot{m}_k & \equiv \text{Axial mass flow rate in subchannel } k \\
\dot{w}_{j,k} & \equiv \text{Pressure driven lateral mass flow rate per unit length from subchannel } j \text{ to subchannel } k (+)
\end{align*}

$$\sum_i \dot{m}_i - \sum_e \dot{m}_e = \dot{m}_k \bigg|_{z} - \dot{m}_k \bigg|_{z+\Delta z} + \sum_j \dot{w}'_{j,k} \Delta z$$  (2)

$$\frac{dM_k}{dt} = \dot{m}_k \bigg|_{z} - \dot{m}_k \bigg|_{z+\Delta z} + \sum_j \dot{w}'_{j,k} \Delta z$$  (3)

For $M_k = \rho_k V_k = \rho_k A_k \Delta z$

$$\dot{m}_k = \rho_k v_k A_k$$

$$A_k \Delta z \frac{d\rho_k}{dt} = \rho_k v_k A_k \bigg|_{z} - \rho_k v_k A_k \bigg|_{z+\Delta z} + \sum_j \dot{w}'_{j,k} \Delta z$$  (4)

To obtain the differential form, divide by $\Delta z$ and take the limit as $\Delta z$ goes to zero

$$A_k \frac{\partial \rho_k}{\partial t} + \frac{\partial}{\partial z} \left( \rho_k v_k A_k \right) = \sum_j \dot{w}'_{j,k}$$  (5)

which is similar to the one-dimensional form considered previously with the addition of the lateral flow terms.

Energy Balance

Consider next an energy balance on control volume $V_k$. As in the one-dimensional case, we again apply the first law to the control volume, and for simplicity we neglect kinetic and potential energy terms

$$\dot{Q}_k + \sum_i \dot{m}_i h_i = \frac{dM_k u_k}{dt} + \sum_e \dot{m}_e h_e$$  (6)

or

$$\frac{dM_k u_k}{dt} + \sum_e \dot{m}_e h_e - \sum_i \dot{m}_i h_i = \dot{Q}_k$$  (7)

The convective terms have three components: a) Energy transport due to axial flow, b) Energy exchange between the subchannels due to pressure driven convective flows and c) Energy exchange between the subchannels due to turbulent mixing flow.
\[
\sum_{c} \dot{m}_c h_c - \sum_{i} \dot{m}_i h_i = \dot{m}_k h_k \bigg|_{z+\Delta z} - \dot{m}_k h_k \bigg|_{z} - \sum_{j} \tilde{w}_{j,k} \Delta z h^* - \sum_{j} \tilde{w}_{j,k} \Delta z \left(h_j - h_k\right)
\]  \tag{8}

where \(h^* = \begin{cases} h_j & \tilde{w}_{j,k} > 0 \\ h_k & \tilde{w}_{j,k} < 0 \end{cases} \)  \tag{9}

The energy equation for \(V_k\) can then be written

\[
\frac{dM_k u_k}{dt} + \dot{m}_k h_k \bigg|_{z+\Delta z} - \dot{m}_k h_k \bigg|_{z} - \sum_{j} \tilde{w}_{j,k} \Delta z h^* - \sum_{j} \tilde{w}_{j,k} \Delta z \left(h_j - h_k\right) = \dot{q}^*_k \Delta z
\]  \tag{10}

where \(\dot{Q}_k = \dot{q}^*_k \Delta z\)

or

\[
V_k \frac{d\rho_k u_k}{dt} + \rho_k v_k h_k A_k \bigg|_{z+\Delta z} - \rho_k v_k h_k A_k \bigg|_{z} - \sum_{j} \tilde{w}_{j,k} \Delta z h^* - \sum_{j} \tilde{w}_{j,k} \Delta z \left(h_j - h_k\right) = \dot{q}^*_k
\]  \tag{11}

We again obtain the differential form by dividing by \(\Delta z\) and taking the limit as \(\Delta z\) goes to zero

\[
A_k \frac{\partial \rho_k u_k}{\partial z} + \frac{\partial}{\partial z} (\rho_k v_k h_k A_k) - \sum_{j} \tilde{w}_{j,k} h^* - \sum_{j} \tilde{w}_{j,k} \left(h_j - h_k\right) = \dot{q}^*_k
\]  \tag{12}

The turbulent mixing term \(\tilde{w}_{j,k}\) is obtained empirically from experimental data and normally correlated in terms of the axial Reynolds number.

**Momentum Equations**

Momentum is a vector quantity, with momentum balances required in both the axial and lateral directions.

**Axial Momentum**

The axial momentum balance gives

\[
\frac{dM_k v_k}{dt} + \sum_{c} \left(\dot{m}_k v_k\right)_c - \sum_{i} \left(\dot{m}_i v_k\right)_i = \sum F_z
\]  \tag{13}

(note the similarity to the energy balance equation). As in the energy balance, the convective terms have three components

\[
\sum_{c} \left(\dot{m}_k v_k\right)_c - \sum_{i} \left(\dot{m}_i v_k\right)_i = \dot{m}_k v_k \bigg|_{z+\Delta z} - \dot{m}_k v_k \bigg|_{z} - \sum_{j} \tilde{w}_{j,k} \Delta z v^* - \sum_{j} \tilde{w}_{j,k} \Delta z \left(v_j - v_k\right)
\]  \tag{14}
where \[ v^* = \begin{cases} v_j & \text{if } \hat{w}^*_{j,k} > 0 \\ v_k & \text{if } \hat{w}^*_{j,k} < 0 \end{cases} \] (15)

The forces acting on the fluid in the axial direction are pressure, viscous (friction) and weight forces

\[ \sum F_z = A_k \left( P_k \bigg|_{z} - P_k \bigg|_{z+\Delta z} \right) - (\tau \rho g) k \Delta z - \rho_k A_k \Delta z \] (16)

where the cross sectional area of the channel has been assumed constant.

The axial momentum equation is then

\[ A_k \Delta \frac{d\rho v_k}{dt} + \hat{m}_k v_k \bigg|_{z+\Delta z} - \hat{m}_k v_k \bigg|_z - \sum_j \hat{w}_{j,k}^* \Delta v^* - \sum_j \hat{w}_{j,k} \Delta (v_j - v_k) = A_k \left( P_k \bigg|_{z} - P_k \bigg|_{z+\Delta z} \right) - (\tau \rho g) k \Delta z - \rho_k A_k \Delta z \] (17)

or in differential form

\[ A_k \frac{\partial \rho v_k}{\partial t} + \frac{\partial}{\partial \Delta z} \left( \rho_k v_k v_k A_k \right) - \sum_j \hat{w}_{j,k}^* v^* - \sum_j \hat{w}_{j,k} (v_j - v_k) = -A_k \frac{\partial P_k}{\partial \Delta z} - (\tau \rho g) k \Delta z - \rho_k A_k \Delta z \] (18)

The result of mixing between channels is a net transfer of mass, energy and momentum between subchannels. This is easily seen by comparing the subchannel form of the fluid equations with the one-dimensional forms considered previously. It should be noted, that in the absence of cross flow, the subchannel equations reduce to the familiar one-dimensional forms. The addition of the cross flow terms implies the need for additional momentum equations in the lateral direction.

**Flow Redistribution**

Variations in hydraulic conditions among subchannels leads to differing axial pressure drops. Therefore, at any axial position there will be lateral pressure gradients leading to lateral cross flows between subchannels. The magnitude of these flows can be determined via a lateral momentum equation. Consider a control volume centered on the gap between subchannels.

![Figure 3: Lateral Momentum Control Volume](image-url)
The momentum equation in the lateral \((y)\) direction is

\[
\frac{dMv_y}{dt} + \sum_e (\dot{m}v_y)_e - \sum_i (\dot{m}v_y)_i = \sum F_y \tag{19}
\]

Note:

\[
M = \rho S \ell \Delta z \\
Mv_y = \rho S \ell \Delta z v_y = (\rho v_y S) \ell \Delta z = \dot{w}_{y,k} \ell \Delta z \\
\sum_e (\dot{m}v_y)_e - \sum_i (\dot{m}v_y)_i = \dot{m}v_y \bigg|_{z+\Delta z} - \dot{m}v_y \bigg|_z + \dot{w}_{y,k} \ell \Delta z v_y \bigg|_k - \dot{w}_{y,k} \ell \Delta z v_y \bigg|_j + \text{turbulent terms} \tag{21}
\]

\[
\dot{m} = \rho v_y S \ell \\
\therefore \dot{m}v_y = \rho v_y S \ell v_y = (\rho v_y S) \ell v_y = \dot{w}_{y,k} \ell v_y \tag{22}
\]

\[
\sum_e (\dot{m}v_y)_e - \sum_i (\dot{m}v_y)_i = \dot{w}_{y,k} \ell v_y \bigg|_{z+\Delta z} - \dot{w}_{y,k} \ell v_y \bigg|_z + \dot{w}_{y,k} \ell \Delta z v_y \bigg|_k - \dot{w}_{y,k} \ell \Delta z v_y \bigg|_j + \text{turbulent terms} \tag{23}
\]

As \(v_z\) will typically be much greater than \(v_y\) we neglect the \(v_y^2\) terms. The turbulent terms are usually neglected as well, such that the lateral momentum equation is

\[
\ell \Delta z \frac{d\dot{w}_{y,k}}{dt} + \dot{w}_{y,k} \ell v_z \bigg|_{z+\Delta z} - \dot{w}_{y,k} \ell v_z \bigg|_z = \sum F_y \tag{24}
\]

The forces acting on the flow in the lateral direction are friction and pressure, i.e.

\[
\sum F_y = S \Delta z \left( P_j - P_k \right) - (\tau_w P_w) y \Delta z \tag{25}
\]

giving for the lateral momentum equation

\[
\ell \Delta z \frac{d\dot{w}_{y,k}}{dt} + \dot{w}_{y,k} \ell v_z \bigg|_{z+\Delta z} - \dot{w}_{y,k} \ell v_z \bigg|_z = S \Delta z \left( P_j - P_k \right) - (\tau_w P_w) y \Delta z \tag{26}
\]

Solution of the subchannel equations provides detailed thermal hydraulic information which is used in determining core thermal limits (e.g. DNB ratios). These solutions must be performed numerically, and as the subchannel equations effectively couple each subchannel in an assembly (or in the core), solutions can be computationally intensive. Examples of computer codes which have been written to solve the subchannel equations include COBRA and VIPRE.