Test 2

This is an open book test. Please, state clearly the theorems you are using, justify your answers and write clearly to get credit for your work.

(1) (3Pts) Let $S \subset \mathbb{R}$ and $f$ be defined on $\mathbb{R}$ by

$$f(x) = \inf \{|x - s| : s \in S\}.$$

Prove that $f$ is uniformly continuous.

For each $s \in S$, let $x, y \in \mathbb{R}$.

$$|x - s| \leq |x - y| + |y - s|$$

Thus

$$\inf_{s \in S} |x - s| \leq |x - y| + \inf_{s \in S} |y - s|$$

Similarly,

$$\inf_{s \in S} |y - s| - \inf_{s \in S} |x - s| \leq |x - y|$$

Thus:

$$f(x) \text{ is Lipschitz in } \mathbb{R},$$

and thus, it is uniformly continuous.

(2) (3Pts) If $r > 0$ a rational number, let

$$g(x) = \begin{cases} 
  x^r \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0
\end{cases}$$

Determine those values of $r$, if any, for which $g'(0)$ exists.

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x} = \lim_{x \to 0} x^{r-1} \sin \frac{1}{x} = 0 \quad \text{provided } r > 1$$

In fact, if $r > 1$, then

$$0 \leq |x^{r-1} \sin \frac{1}{x}| \leq |x^{r-1}|$$

and we can use the Squeeze Theorem.

This shows that $g'(0)$ exists for $r > 1$. 
(3) (3Pts) Give an example of:

(a) A function $f$ whose derivative exist for all $x \in \mathbb{R}$, but such that $f'(x)$ is discontinuous at one point.

$$f(x) = \begin{cases} 
x^2 \sin \frac{1}{x} & x \neq 0 \\
0 & x = 0 \end{cases}$$

$$f'(x) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$  

$f'$ is discontinuous at $x = 0$.

(b) A function $f$ strictly increasing whose derivative vanishes at one point.

$$f(x) = x^3$$

is strictly increasing.

$$f'(x) = 3x^2 = 0 \quad \forall \quad x = 0$$

(4) (3Pts) Let $f$ be a twice differentiable function on $(a, b)$ and let there be points $x_1 < x_2 < x_3$ in $(a, b)$ such that $f(x_1) > f(x_2)$ and $f(x_2) < f(x_3)$. Prove that there is point $c \in (a, b)$ such that $f''(c) > 0$ (Hint: Mean Value Theorem).

By MVT

$$f(x_2) - f(x_1) = f'(c), \quad c \in (x_1, x_1)$$

Since $f(x_1) < f(x_1)$ and $x_2 > x_1$,

$$f'(c) < 0$$

Similarly,

$$f(x_3) - f(x_2) = f'(d), \quad d \in (x_2, x_3)$$

Since $f'(d) > 0$ since $f(x_3) > f(x_2), x_3 > x_2$.

Since $f'$ is differentiable, we can use MVT on $f'$ on yet another interval.

$$f'(d) - f'(c) = f''(e), \quad e \in (c, d)$$

Since $f'(d) > 0, f'(c) < 0$, then $f''(e) > 0$

Necessarily $(c, d) \subseteq (a, b)$, so

Thus $e \in (a, b)$. 

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