Test 2

This is an open book test. Please, state clearly the theorems you are using, justify your answers and write clearly to get credit for your work.

(1) (3 Pts) Let $F : D \rightarrow \mathbb{R}$ be continuous at $c \in D$. Prove that there is a constant $M > 0$ and a neighborhood $V$ of $c$ such that $|f(x)| \leq M$ for all $x \in V \cap D$.

Since $f$ is continuous at $c$, there is a $\delta > 0$ s.t.

$$|f(x) - f(c)| < 1 \quad \text{for all} \quad |x - c| < \delta$$

Thus

$$-1 < f(x) - f(c) < 1$$

and

$$f(c) - f(x) < 1 + |f(c)|$$

for all $x \in V$. Let

$M = 1 + |f(c)|$

Then

$$|f(x)| \leq 1 + |f(c)| \quad \forall x \in V \delta(c)$$

(3) (3 Pts) Suppose that $f : I \in \mathbb{R}$, where $I$ is an interval, is differentiable on $I$, and $f'$ is bounded on $I$. Prove that $f$ is a Lipschitz function on $I$.

By assumption, $|f'(x)| \leq M \quad \forall x \in I$, where $M > 0$.

By the Mean Value Theorem, for each $x, y \in I$, $x \neq y$ there is a $c \in (x, y)$ s.t.

$$f(y) - f(x) = f'(c)(y - x)$$

Thus

$$|f(y) - f(x)| = |f'(c)||y - x|$$

Since $|f'(c)| \leq M$, then

$$|f(y) - f(x)| \leq M |y - x| \quad \forall x, y \in I$$

This shows that $f$ is a Lipschitz function on $I$. 
(2) (5 pts) Suppose that $f$ is continuous on $[a, b]$ and $\int_a^b f g = 0$ for every integrable function $g \in \mathcal{R}[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Arguing by contradiction, suppose that there is a $c \in (a, b)$ s.t. $f(c) \neq 0$. (*)

Without loss of generality, suppose that $f(c) > 0$.

Then, $\exists \delta > 0$ s.t. $f(x) > 0$.

In fact, since $f$ is continuous at $c$, given $\epsilon = \frac{1}{2} f(c)$ there exists $\delta > 0$ s.t. if $x \in (c - \delta, c + \delta)$, then

$$|f(x) - f(c)| < \frac{1}{2} f(c)$$

Thus,

$$f(x) > \frac{1}{2} f(c), \quad \forall x \in (c - \delta, c + \delta)$$

Define $g(x) = \begin{cases} \frac{1}{2\delta} & \text{if } x \in (c - \delta, c + \delta) \\ 0 & \text{if } x \notin (c - \delta, c + \delta) \end{cases}$

$g \in \mathcal{R}[a, b]$ since $g$ is a step function.

Then

$$\int_a^b f g = \int_{c-\delta}^{c+\delta} f(x) \frac{1}{2\delta} dx > \frac{2\delta}{2\delta} \frac{1}{2} f(c) = \frac{1}{2} f(c) > 0$$

This is a contradiction. Thus $f = 0 \quad \forall x \in [a, b]$.

(*) The argument can be easily adjusted to the case where $c = a$ or $c = b$. 

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(4) (5 Pts) Let \( f, g : D \to \mathbb{R} \). Show that:

(a) if \( f, g \) are uniformly continuous on \( D \) and bounded on \( D \), then \( fg \) is uniformly continuous;

(b) if \( f, g \) are uniformly continuous on \( D \) and \( D \) is a bounded set, then \( fg \) is uniformly continuous (Hint: use part (a)).

(a) Since \( f, g \) are uniformly continuous, then, given any \( \varepsilon > 0 \),

there is a \( \delta > 0 \) (\( \delta = \delta(\varepsilon) \), but \( \delta \) does not depend on \( x, y \))

such that if \( |x - y| < \delta \), then

\[ |f(x) - f(y)| < \varepsilon, \quad |g(x) - g(y)| < \varepsilon \]

Observe that

\[ |f(x)g(x) - f(y)g(y)| \leq |f(x)g(x) - f(x)g(y)| + \]

\[ + |f(x)g(y) - f(y)g(y)| \]

\[ \leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \]

Since \( f, g \) are bounded on \( D \), then \( |f(x)||g(x)| \leq M, \quad \forall x \in D \)

Thus, for all \( |x - y| < \delta \),

\[ |f(x)g(x) - f(y)g(y)| \leq M \varepsilon + M \varepsilon = 2M \varepsilon \]

This shows that \( fg \) is uniformly continuous on \( D \).

(b) If \( f, g \) are uniformly continuous on \( D \) and \( D \) is bounded,

then, by a result proved in class, \( f, g \) are bounded on \( D \)

thus the uniform continuity of \( fg \) follows from part (a).