(1) (3Pts) Find all $x \in \mathbb{R}$ that satisfy $|x + 1| < |2x - 1|.$

\[
\begin{align*}
\text{If } x < -1 & : & -x - 1 < -2x + 1 & \Rightarrow & & \text{x < 2} \\
\text{If } x < \frac{1}{2} & : & x + 1 < -2x + 1 & \Rightarrow & & 3x < 0 & \quad (x < 0) \\
\text{If } x > \frac{1}{2} & : & x + 1 < 2x - 1 & \Rightarrow & & x > 2
\end{align*}
\]

The inequality is satisfied for:

\[x < 0 \quad \text{and} \quad x > 2\]

(2) (4Pts) Give an example of:

(a) A bounded sequence that does not converge.

\[(-1)^n \quad |(-1)^n| \leq 1\]

(b) Two divergent sequences whose sum converges.

\[a_n = (-1)^n \quad b_n = (-1)^{n+1} \quad a_n + b_n = (-1)^n + (-1)^{n+1} = 0 \quad \forall n \geq 0\]

(c) A divergent sequence \((x_n)\) for which the sequence of absolute values \(|x_n|\) is convergent.

\[a_n = (-1)^n \quad |a_n| = 1\]

(d) A divergent sequence with a convergent subsequence (state both the sequence and the subsequence).

\[a_n = (-1)^n \quad a_{2n} = (-1)^{2n} = 1\]
(3) (3Pts) Let \( a_1 = 1, a_{n+1} = \sqrt{2a_n + 3}, n \in \mathbb{N} \). Prove that \( (x_n) \) is convergent and find the limit. (Hint: use Monotone Convergence Theorem).

(\( a_n \) is bounded: (I) \( a_1 \leq 3 \)
(II) Suppose \( a_n \leq 3 \)
(III) Then \( a_{n+1} = \sqrt{2a_n + 3} \geq \sqrt{2 \cdot 3 + 3} = 3 \)

(\( a_n \) is increasing: (I) \( a_1 = 1 \leq a_2 = \sqrt{3} \)
(II) Suppose \( a_n \leq a_{n+1} \)
(III) Then \( a_{n+2} = \sqrt{2a_{n+1} + 3} \geq \sqrt{2a_n + 3} = a_{n+1} \)

By the Monotone Convergence Theorem, \( (a_n) \) is convergent.

Thus: \( \lim a_{n+1} = \lim \sqrt{2a_n + 3} \). Let \( \lim a_n = L \)

\[
L = \sqrt{2L + 3} \Rightarrow L^2 - 2L - 3 = 0 \quad (L = 3) \quad \text{or} \quad L = -1
\]

The limit is \( L = 3 \) \( (L = -1 \text{ is not acceptable since } a_1) \).

(4) (2Pts) Compute the limit of \( (1 + \frac{1}{n})^{n/2} \). (State the properties and/or theorems you use when you deduce your result).

We know that \( \lim_{n \to \infty} (1 + \frac{1}{n})^n = e \)

We also know that if \( a_n \geq 0 \) and \( \lim a_n = L \),

then \( \sqrt[n]{a_n} \) converges and \( \lim \sqrt[n]{a_n} = \sqrt[L]{} \).

Thus \( \lim (1 + \frac{1}{n})^{n/2} = \left( \lim \left(1 + \frac{1}{n}\right)^n \right)^{1/2} = e^{1/2} \).
(5) (3Pts) Show directly from the definition that if \((x_n)\) and \((y_n)\) are Cauchy sequences, the \((x_n y_n)\) is also a Cauchy sequence. Justify carefully all your steps.

Since \((x_n), (y_n)\) are Cauchy, then they are bounded.

Let \(|x_n| \leq M, \ |y_n| \leq M \ \forall n\)

Since \((x_n), (y_n)\) are Cauchy, then given any \(\varepsilon > 0\),

\[\exists N \text{ s.t.} \ \ |x_n - x_m| < \frac{\varepsilon}{2M} \]

\[|y_n - y_m| < \frac{\varepsilon}{2M} \ \forall n,m > N\]

It follows that

\[|x_n y_n - x_m y_m| \leq |x_n y_n - x_n y_m + x_n y_m - x_m y_m|\]

\[\leq |x_n||y_n - y_m| + |y_m||x_n - x_m|\]

\[< M \frac{\varepsilon}{2M} + M \frac{\varepsilon}{2M} < \varepsilon \ \forall n,m > N\]

Thus \((x_n y_n)\) is a Cauchy sequence.