The Fundamental Theorem of Calculus

Let \( F : [a, b] \to \mathbb{R} \) s.t.

(a) \( F \) is continuous on \([a, b]\)

(b) \( F'(x) = f(x) \quad \forall x \in (a, b) \)

(c) \( f \in R(a, b) \)

Then

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

Fix \( a \leq x \leq b \). Since \( F \in R([a, b]) \), we \( \exists \delta > 0 \) s.t. \( F \) is a tagged partition of \([a, b]\)

Thus:

\[
F(b) - F(a) = \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = \sum_{i=1}^{n} f(c_i) \cdot (x_i - x_{i-1})
\]

If we choose \( t = 0 \) in the dirivative \( F' \), then

\[
\left| F(b) - F(a) \right| = \int_a^b f(x) \, dx
\]

\( \overset{\text{R.H.}}{=} \) \( \overset{\text{L.H.}}{=} \) \( \overset{\text{Theorem}}{=} \)

If \( F \) is differentiable, \( \text{L.H.} \) is satisfied

If \( F \) is differentiable, \( \text{R.H.} \) is not automatically satisfied

**Def** If \( F \) is a function satisfying the assumption of the theorem, then \( F \)

is called the antiderivative or primitive of \( f \).

\( \text{Ex} \quad (a) \quad F(x) = \frac{x^2}{2} \quad x \in [a, b] \)

Thus, all assumptions of FTC are satisfied and

\[
\int_a^b f(x) \, dx = \left[ \frac{x^2}{2} \right]_a^b = \frac{b^2 - a^2}{2}
\]

\( \text{Ex} \quad (b) \quad g(x) = \begin{cases} x & x \in [-10, 10] \\ 0 & \text{otherwise} \end{cases} \)

\[
\int_{-10}^{10} g(x) \, dx = \begin{cases} 10 - (-10) = 20 & x \in [-10, 10] \\ 0 & \text{otherwise} \end{cases}
\]

\( \text{Ex} \quad (c) \quad h(x) = \begin{cases} x & x \in [0, b] \\ 0 & \text{otherwise} \end{cases} \)

It is continuous on \([0, b]\) and \( H'(x) = 1/x \quad \forall x \in (0, b) \). Since \( h \neq H' \) is not bounded on \((0, b)\), then \( F \) does not belong to \( R(a, b) \).

Thus, FTC does not apply. (\( h \neq H' \) is not a generalized Riemann integrable on \((0, b)\)).
Def. If $f \in R([a,b])$, then the function

$$F(x) = \int_a^x f(t) \, dt, \quad x \in [a,b]$$

is the **indefinite integral** of $f$ with basepoint $a$.

Thus, the indefinite integral $F(x)$ is continuous on $[a,b]$.

In fact, if $f(x) \leq F(x)$, then $|F(x) - F(w)| \leq M |x - w| \quad \forall x, w \in [a,b]$

**Proof.** In $[a,b]$, we have

$$F(x) = \int_a^x f(t) \, dt = \int_a^w f(t) \, dt + \int_w^x f(t) \, dt = F(w) + \int_w^x f(t) \, dt$$

Thus, $F(x) - F(w) = \int_w^x f(t) \, dt$$

Since $|f(t)| \leq M$ in $[a,b]$, then $-M |x - w| \leq \int_w^x f(t) \, dt \leq M |x - w|$

Hence, $|F(x) - F(w)| \leq M |x - w|$

Thus (FTC, second form) let $f \in R([a,b])$ and $F$ be the function at $c \in (a,b)$.

Then $F(x) = \int_a^x f(t) \, dt$ is differentiable at $c$ and $F'(c) = f(c)$.

**Proof.** Let $c \in (a,b)$ and choose the right-hand derivative of $F$ at $c$.

Since $f$ is continuous at $c$, given $\delta > 0$, $\exists \eta > 0$ s.t.

$$0 < x - c < \eta \Rightarrow \int_c^{c+\eta} f(t) \, dt < \eta$$

Choose $k$ satisfying $0 < k < \eta$. Then

$$F(c + k) - F(c) = \int_c^{c+k} f(t) \, dt$$

Using inequality (i), we have

$$\left| F(c + k) - F(c) \right| \leq \int_c^{c+k} |f(t) - f(c)| \, dt \leq M |x - c| + \int_c^{c+k} f(t) \, dt$$

Thus:

$$f(c) = \frac{F(c + k) - F(c)}{k} \rightarrow f(c) \quad k \rightarrow 0$$

This shows that

$$\lim_{k \rightarrow 0} \frac{F(c + k) - F(c)}{k} = f(c)$$

The left-hand limit is computed in the same way.

**Corollary.** If $f$ is continuous on $[a,b]$, then $F(x) = \int_a^x f(t) \, dt$ is differentiable on $[a,b]$ and $F'(x) = f(x)$. 


Ex. Let \( f : (-1,1) \to \mathbb{R} \) be a function and let \( F(x) = \int_0^x f(t) \, dt \). Show that if \( F(x) \) is differentiable at some point \( x_0 \) in \((-1,1)\), then \( F(x) \) is differentiable on \((-1,1)\).

**Proof.**

Assume \( F(x) \) is differentiable at \( x_0 \). Then, by the Fundamental Theorem of Calculus, we have

\[
F(x) = \left. \frac{d}{dx} \int_0^x f(t) \, dt \right|_{x=x_0} = f(x_0)
\]

for every \( x \) in \((-1,1)\) such that \( x \neq x_0 \). Therefore, \( F(x) \) is differentiable on \((-1,1)\).

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Let \( f : [a,b] \to \mathbb{R} \) be a bounded function. According to Lebesgue's Integrability Criterion, \( f \) is integrable on \([a,b] \) if it is continuous \( \forall x \in (a,b) \setminus N \), where \( N \) is a null set.

- Lebesgue's Integrability Criterion: A bounded function \( f : [a,b] \to \mathbb{R} \) is integrable on \([a,b] \) if it is continuous \( \forall x \in (a,b) \setminus N \), where \( N \) is a null set.

Let \( f : [a,b] \to \mathbb{R} \) be a bounded function and let \( g : [a,b] \to \mathbb{R} \) be a continuous function. Then, the composition \( g \circ f : [a,b] \to \mathbb{R} \) is integrable on \([a,b] \).

**Proof.**

By the Lebesgue's Integrability Criterion, \( f \) is integrable on \([a,b] \). Therefore, \( (g \circ f) \) is also integrable on \([a,b] \).