Problem 1. Let $>$ be a total order on the monomials in $k[x_1, \ldots, x_n]$ that satisfies the multiplicative property (i.e. $x^\alpha > x^\beta$ implies that $x^\alpha x^\gamma > x^\beta x^\gamma$). Show that $>$ is a well-ordering if and only if for every $\alpha \in \mathbb{Z}^n_{\geq 0}$, $1 \leq x^\alpha$.

Problem 2. Fix a monomial ordering $>$ on $k[x_1, \ldots, x_n]$ and suppose that $\{f_1, \ldots, f_n\}$ is a Gröbner basis for an ideal $I$ with the property that for every $i = 1, \ldots, n$, the leading term $LT(f_i)$ is $x_i^{d_i}$ for some $d_i \in \mathbb{Z}_{\geq 0}$. Show that $k[x_1, \ldots, x_n]/I$ is a finite-dimensional vectorspace and find a formula for its dimension in terms of $d_1, \ldots, d_n$.

Problem 3. Let $R$ be the ring of functions $f : \mathbb{R} \to \mathbb{R}$, and define $J \subset R$ to be the set of functions that vanish on some open neighborhood of 0:

$$J = \{f \in R \text{ such that there exists } \epsilon > 0 \text{ with } f(a) = 0 \text{ for all } a \in (-\epsilon, \epsilon)\}.$$

(a) Show that $J$ is an ideal of the ring $R$.
(b) Show that $J$ is not finitely generated.
(c) Define an ascending chain of ideals $I_1 \subset I_2 \subset I_3 \subset \ldots$ in $J$ that does not terminate.

Problem 4. Hassett 2.12