1. (35 points) Consider the linear functional $f$ on $\mathbb{R}^3$ defined by $f(\alpha) = \gamma \cdot \alpha$ (dot product of vectors) where $\gamma = (1, 0, -1)$. Let $W$ be the kernel of $f$. Find a linear operator $T$ on $\mathbb{R}^3$ such that $T\alpha = \alpha$ for $\alpha \in W$ and $T\gamma = -\gamma$. Determine $T(x_1, x_2, x_3)$.

2. (35 points) Consider the real vector space $V$ of polynomials from $\mathbb{R}[x]$ of degree $\leq 2$ and the linear operator $T$ given by $(Tp)(x) = xp'(x)$ for $p \in V$. Let $f_1, f_2, f_3$ be the linear functionals on $V$ defined by $f_1(p) = p(0), \ f_2(p) = p'(0), \ f_3(p) = \int_0^1 p(x)dx, \ p \in V.$

Find the matrix of $T$ relative to the basis $\{p_1, p_2, p_3\}$ for $V$, whose dual basis is $\{f_1, f_2, f_3\}$.

3. (30 points) Let $A$ be a linear operator on a finite-dimensional vector space $V$. A subspace $W$ of $V$ is called $A$-invariant if $A(\alpha) \in W$ for all $\alpha \in W$. Prove that $W$ is $A$-invariant if and only if $W^0$ is $A'$-invariant.