1. Let $V$ be the vector space of all $2 \times 2$ matrices with entries in a field $F$. Let $W_1$ be the subset of all matrices of the form $\begin{bmatrix} a & b \\ 2b & a \end{bmatrix}$, and $W_2$ be the subset of all matrices of the form $\begin{bmatrix} a & -a \\ b & c \end{bmatrix}$.
   
   a. Show that $W_1$ and $W_2$ are subspaces of $V$.
   b. Find bases for $W_1$, $W_2$ and $W_1 \cap W_2$.
   c. Find the dimensions of $W_1$, $W_2$, $W_1 \cap W_2$ and $W_1 + W_2$.

2. Let $V$ be the vector space of all real-valued polynomials $p(x)$ of degree $\leq 2$, and $T: V \to \mathbb{R}^2$ be the map given by
   
   $$T(p) = \left( p'(1), \int_0^1 p(x) \, dx \right).$$
   
   a. Check that $T$ is a linear transformation.
   b. Find a basis for $\text{Ker} T$ and determine the rank of $T$.
   c. Find the matrix of $T$ relative to the pair of ordered bases $\{1, x, x^2\}$, $\{(1, 1), (0, 1)\}$.

3. Let $T$ be the linear operator on $\mathbb{R}^3$ represented by the matrix
   $$\begin{bmatrix} 0 & 1 & -1 \\ -1 & -2 & 5 \\ 0 & 0 & 3 \end{bmatrix}.$$
   
   a. Find the characteristic polynomial and the minimal polynomial of $T$.
   b. Determine all eigenvalues and eigenvectors of $T$. Is $T$ diagonalizable?
   c. Find two nonzero $T$-invariant subspaces $W_1$ and $W_2$ of $\mathbb{R}^3$ such that $\mathbb{R}^3 = W_1 \oplus W_2$.
   d. Find the Jordan form of $T$ a basis for $\mathbb{R}^3$ in which $T$ is in Jordan form.

4. For fixed $c \in \mathbb{R}$, $n \in \mathbb{N}$, denote by $V$ the vector space of all real-valued functions of the form $p(x)e^{cx}$ where $p(x)$ is a polynomial of degree $\leq n - 1$. Note that $V$ is invariant under $D = d/dx$, and consider $D$ as a linear operator on $V$.
   
   a. Find the characteristic polynomial and the minimal polynomial of $D$.
   b. Prove that $V$ has a cyclic vector for $D$.
   c. Prove that every linear operator $T$ on $V$ that commutes with $D$ is a polynomial of $D$.

5. Check that $f(A, B) = \text{tr}(AB)$ is a non-degenerate symmetric bilinear form on the vector space of $n \times n$ matrices with entries in $\mathbb{C}$.