2. For \( \forall \alpha = (x_1, x_2, x_3) \in \mathbb{R}^3 \) we have 
\[
T\alpha = (2x_1, 2x_2, -x_3),
\]
\[
T^2\alpha = (4x_1, 4x_2, x_3) = 2\alpha + T\alpha \in \text{span}\{\alpha, T\alpha\}.
\]
Then by induction, \( T^k\alpha \in \text{span}\{\alpha, T\alpha\} \) \( \forall k \geq 0 \)
\[\Rightarrow Z(\alpha; T) = \text{span}\{\alpha, T\alpha\} \neq \mathbb{R}^3.\]
Hence, \( T \) has no cyclic vectors.

Now let \( \alpha = (1, -1, 3) \). Then, to find a basis of 
\( Z(\alpha; T) = \text{span}\{ (1, -1, 3), (2, -2, -3) \} \)
we row reduce
\[
\begin{bmatrix}
1 & -1 & 3 \\
2 & -2 & -3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 3 \\
0 & 0 & -9
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 3 \\
0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[\Rightarrow Z(\alpha; T) \text{ has a basis } \{ (1, -1, 0), (0, 0, 1) \} .\]

5. First, note that \( N^n = 0 \) (see Problem 15, Sec. 6.8).
Then if \( N^{n-1}\alpha \neq 0 \Rightarrow \) the \( T \)-annihilator \( p_\alpha \) of \( \alpha \)
is not \( x^{n-1} \) and divides \( x^n \) \( \Rightarrow p_\alpha(x) = x^n. \)
Then \( \dim Z(\alpha; T) = \deg p_\alpha = n = \dim V \)
\[\Rightarrow V = Z(\alpha; T), \text{ and } \alpha \text{ is a cyclic vector.}\]
One can also check directly that the vectors \( \alpha, N\alpha, \ldots, N^{n-1}\alpha \) are linearly independent. Otherwise, \( g(T)\alpha = 0 \) for some polyn. \( g(x) \) with \( \deg g \leq n-1 \), but \( g(x) | x^n \implies g(x) = x^k \), \( k \leq n-1 \implies T^{n-1}\alpha = 0 \) — contradiction.

Hence, \( \{ \alpha, N\alpha, \ldots, N^{n-1}\alpha \} \) is a basis of \( V \).

The matrix of \( N \) in this basis is

\[
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
\end{bmatrix}
\]

This is the companion matrix of the polyn. \( x^r \).

7 (a) \( T \)-diagonalizable \( \Rightarrow \) min. polyn. is \( p(x) = (x - c_1) \cdots (x - c_k) \) with distinct \( c_1, \ldots, c_k \).

If \( T \) has a cyclic vector, then \( p(x) = f(x) \) (where \( f \) is the characteristic polyn.)

\( \Rightarrow k = \deg p = n \)

\( \Rightarrow T \) has \( n \) distinct eigenvalues.
(b) We are given \( \alpha = \alpha_1 + \ldots + \alpha_n \), \( T \alpha_i = c_i \alpha_i \)
and \( c_i \neq c_j \) for \( i \neq j \). Then
\[
g(T) \alpha = g(T) \alpha_1 + \ldots + g(T) \alpha_n
= g(c_1) \alpha_1 + \ldots + g(c_n) \alpha_n.
\]
If \( g(T) \alpha = 0 \), then \( g(c_1) = \ldots = g(c_n) = 0 \)
\( \Rightarrow \) \( g(x) \) is divisible by \( p(x) = (x-c_1) \ldots (x-c_n) \).
\( \Rightarrow \) the \( T \)-annihilator \( p_\alpha(x) = p(x) \).

Then \( \dim \mathbb{Z}(\alpha; T) = \deg p_\alpha = n = \dim V \)
\( \Rightarrow \mathbb{Z}(\alpha; T) = V \) \( \Rightarrow \alpha \) is a cyclic vector.

8. Let \( \alpha \) be a cyclic vector; then \( V \) has a basis \( \{ \alpha, T \alpha, \ldots, T^{n-1} \alpha \} \). Hence a vector in \( V \) is of the form \( g(T) \alpha \) for some polyn. \( g(x) \) of \( \deg g \leq n-1 \). If \( U \alpha = g(T) \alpha \), we claim that \( U = g(T) \). Indeed,
\[
U T^i \alpha = T^i U \alpha = T^i g(T) \alpha = g(T) T^i \alpha.
\]
(because \( U \) and \( T \) commute)

Since \( U = g(T) \) on a basis vector \( T^i \alpha \)
\( \Rightarrow \) \( U = g(T) \).