1) Let $R = \text{im} E$, $N = \ker E$, then for $x \in V$

\[ E\alpha = 0 \iff \alpha \in N, \]
\[ E\alpha = \alpha \iff \alpha \in R. \]

Since $V = R \oplus N$, a linear operator $S$ on $V$ is uniquely determined by its restrictions to $R$ and $N$.

1) Since $E|_N = 0$, both $E|_N$ and $TE$ are $0$ on $N$. Then $E|_N = TE$ on $V \iff$ on $R$. For $x \in R$ we have $E\alpha = \alpha$ and

\[ ETE\alpha = T\alpha \iff T\alpha \in R. \]

Hence, $ET = TE \iff R$ is $T$-invariant.

2) Similarly, $ET = TE$ on $V \iff$

\[ \left\{ \begin{array}{l}
ET\alpha = T\alpha \quad \forall \alpha \in R \\
\text{and} \quad ET\alpha = T\alpha \quad \forall \alpha \in N.
\end{array} \right. \]

For $x \in R$, we have $E\alpha = \alpha$ and $ET\alpha = T\alpha = T\alpha \iff T\alpha \in R$.

For $x \in N$, we have $E\alpha = 0$ and $ET\alpha = T\alpha = 0 \iff T\alpha \in N$.

Thus, $TE = ET \iff$ both $R$ and $N$ are $T$-invariant.
(2) (a) One checks that $T \varepsilon_1 = 2 \varepsilon_1$, 
\[ T \varepsilon_1 \in W_1 \quad \forall \varepsilon_1 \in W_1 = \text{span} \{ \varepsilon_1 \}. \]

(b) Assume $R^2 = W_1 \oplus W_2$. Then $\dim W_2 = 1$ 
\[ W_2 = \text{span} \{ \varepsilon_2 \}. \]
Since $W_1 \cap W_2 = \{0\}$ 
\[ \varepsilon_2 \in \text{span} \{ \varepsilon_1 \}. \]
Now $T \varepsilon_2 \in W_2 = \text{span} \{ \varepsilon_2 \}$ 
\[ T \varepsilon_2 = \lambda \varepsilon_2 \Rightarrow \lambda \varepsilon_2 \text{ is an eigenvector.} \]
But the char. polyn. $f(x) = \det \begin{bmatrix} x-2 & -1 \\ 0 & x-2 \end{bmatrix} = (x-2)^2$
and $\ker \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \text{span} \{ \varepsilon_1 \} \Rightarrow$ all eigenvectors are scalar multiples of $\varepsilon_1$. Contradiction.

(4) For simplicity, assume $k=2$. The general case is similar, or it can be deduced from the case $k=2$ by induction, using that 
\[ W_1 \oplus \cdots \oplus W_k = (W_1 \oplus \cdots \oplus W_{k-1}) \oplus W_k. \]

So, let $V = W_1 \oplus W_2$, $T_i = T |_{W_i}$, and let $B_i$ be a basis of $W_i$ $(i=1,2)$. Then the disjoint union $B = B_1 \cup B_2$ is a basis of $V$ and the matrix of $T$ in the basis $B$ is 
\[ A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \] where $A_i$ is the matrix of $T_i$ w.r.t. the basis $B_i$. 

(a) \( \det T = \det A = \det A_1 \cdot \det A_2 = \det T_1 \cdot \det T_2 \).

(b) \[ f(x) = \det(xI-T) = \det(xI-T_1) \cdot \det(xI-T_2) \]
   \[ = f_1(x) \cdot f_2(x) \quad \text{by part (a)}. \]

(c) Recall that the min. polyn. \( p(x) \) for \( T \) is the unique monic polyn. of min degree s.t. \( p(T) = 0 \) \( (\iff p(A) = 0) \). Moreover, \( g(A) = 0 \iff p \mid g \).

On the other hand,
\[ g(A) = \begin{bmatrix} g(A_1) & 0 \\ 0 & g(A_2) \end{bmatrix} = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]
\[ \iff \begin{cases} g(A_1) = 0 \\ g(A_2) = 0 \end{cases} \iff p \mid g \text{ and } p_2 \mid g. \]

Thus, \( p \) is the monic polyn. of min degree divisible by both \( p_1 \) and \( p_2 \)
\[ \Rightarrow p = \text{least common multiple of } p_1, p_2. \]
9. (a) \( \forall f \in V \) we have
\[
f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}
\]
\[
\implies V = W_e + W_o.
\]
If \( f \in W_e \cap W_o \) \( \implies f(x) = f(-x) = -f(x) \)
\( \implies f(x) = 0 \ \forall x \implies f = 0. \)
Hence, \( V = W_e \oplus W_o. \)

(b) \( 1 \in W_e \) but \( (T1)(x) = x \in W_o. \)
\( x \in W_o \) but \( (Tx)(x) = \frac{x^2}{2} \in W_e. \)
\( \implies \) neither \( W_e \) nor \( W_o \) is \( T \)-invariant.