1. Choose a basis \( B_1 = \{ a_1, \ldots, a_m \} \) of \( W_1 \) and extend it to a basis \( B = \{ a_1, \ldots, a_n \} \) of \( V \). Let \( W_2 = \text{span} \{ a_{m+1}, \ldots, a_n \} \).

Then \( B_2 = \{ a_{m+1}, \ldots, a_n \} \) is a basis of \( W_2 \), because as a subset of the basis \( B \) it's linearly independent. The union \( B = B_1 \cup B_2 \) is disjoint and is a basis of \( V \Rightarrow V = W_1 \oplus W_2 \).

2. Choose bases \( B_i \) of \( W_i \). Then, since \( V = W_1 + \cdots + W_k \), \( \forall \alpha \in V \) can be written as \( \alpha = a_1 + \cdots + a_k \) for some \( a_i \in W_i \Rightarrow \) the set \( B = B_1 \cup \cdots \cup B_k \) spans \( V \). But \( |B| \leq |B_1| + \cdots + |B_k| = \dim W_1 + \cdots + \dim W_k = \dim V \Rightarrow B \) is a basis of \( V \) and \( |B| = |B_1| + \cdots + |B_k| \).

Then the union \( B_1 \cup \cdots \cup B_k \) is disjoint, and hence \( V = W_1 \oplus \cdots \oplus W_k \).
3. One can check that \( \{ \mathbf{a}_1 = (1, -1), \mathbf{a}_2 = (1, 2) \} \)

is a basis of \( \mathbb{R}^2 \). We need a linear operator \( \mathbf{E} \) on \( \mathbb{R}^2 \) s.t. \( \mathbf{E}\mathbf{a}_1 = \mathbf{a}_1, \mathbf{E}\mathbf{a}_2 = 0 \). Let

\( \{ \mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1) \} \)

be the standard basis of \( \mathbb{R}^2 \). Then \( \mathbf{e}_1 = \frac{1}{3} (2 \mathbf{a}_1 + \mathbf{a}_2), \mathbf{e}_2 = \frac{1}{3} (-\mathbf{a}_1 + 2 \mathbf{a}_2) \)

\( \Rightarrow \mathbf{E}\mathbf{e}_1 = \frac{2}{3} \mathbf{E}\mathbf{a}_1 + \frac{1}{3} \mathbf{E}\mathbf{a}_2 = \frac{2}{3} \mathbf{a}_1 = \left( \frac{2}{3}, -\frac{2}{3} \right), \)

\( \mathbf{E}\mathbf{e}_2 = -\frac{1}{3} \mathbf{E}\mathbf{a}_1 + \frac{1}{3} \mathbf{E}\mathbf{a}_2 = -\frac{1}{3} \mathbf{a}_1 = \left( -\frac{1}{3}, \frac{1}{3} \right) \)

\( \Rightarrow \mathbf{E}(x_1, x_2) = \mathbf{E}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = \left( \frac{2}{3} x_1 - \frac{1}{3} x_2, -\frac{2}{3} x_1 + \frac{1}{3} x_2 \right). \)

4. This is false. Consider, for example, two projections \( \mathbf{E}_1, \mathbf{E}_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined by:

\( \mathbf{E}_1(1, 0) = (1, 0) \quad \mathbf{E}_2(0, 1) = (0, 1) \)

\( \mathbf{E}_1(0, 1) = (0, 0) \quad \mathbf{E}_2(1, 1) = (0, 0) \).

Then \( \mathbf{E}_2(1, 0) = \mathbf{E}_2((1, 1) - (0, 1)) = (0, -1) \) and

\( (\mathbf{E}_1 + \mathbf{E}_2)(1, 0) = (1, -1) \),

\( (\mathbf{E}_1 + \mathbf{E}_2)(0, 1) = (0, 1) \)

\( \Rightarrow (\mathbf{E}_1 + \mathbf{E}_2)^2(1, 0) = (\mathbf{E}_1 + \mathbf{E}_2)(1, -1) = (1, -1) - (0, 1)

= (1, -2) \neq (1, -1) = (\mathbf{E}_1 + \mathbf{E}_2)(1, 0) \)

\( \Rightarrow (\mathbf{E}_1 + \mathbf{E}_2)^2 \neq \mathbf{E}_1 + \mathbf{E}_2. \)
5. Write \( f(x) = a_0 + a_1 x + \ldots + a_m x^m \).

Then \( E^2 = E \Rightarrow E^3 = E^2 \cdot E = EE = E \)

\( \Rightarrow E^4 = E^3 \cdot E = EE = E \Rightarrow \ldots \Rightarrow E^m = E \ \forall m \geq 1 \).

Then \( f(E) = a_0 I + a_1 E + a_2 E^2 + \ldots + a_m E^m \)

\[ = a_0 I + (a_1 + a_2 + \ldots + a_m) E. \]

6. True. Choose a basis \( \{ \alpha_1, \ldots, \alpha_n \} \) in which

\( T \) is diagonal: \( T \alpha_i = t_i \alpha_i \) where \( t_i = 0 \) or \( 1 \).

We can reorder the basis so that

\[ T \alpha_i = \alpha_i \quad (i=1, \ldots, m) \]

\[ T \alpha_i = 0 \quad (i=m+1, \ldots, n) \]

for some \( 0 \leq m \leq n \). Let \( W_1 = \text{span} \{ \alpha_1, \ldots, \alpha_m \} \),

\( W_2 = \text{span} \{ \alpha_{m+1}, \ldots, \alpha_n \} \) \Rightarrow \( V = W_1 \oplus W_2 \)

(see Problem 1). We have \( T_{W_1} = I_{W_1} \), \( T_{W_2} = 0 \)

\( \Rightarrow T \) is the projection onto \( W_1 \) along \( W_2 \).

7. We have \( V = R \oplus N \), \( E_R = I_R \), \( E_N = 0 \)

\( \Rightarrow (I-E)_R = 0 \), \( (I-E)_N = I_N \)

\( \Rightarrow I-E \) is the projection onto \( N \) along \( R \).
(a) Multiply the eq. \( E_1 + \ldots + E_k = I \)
on the left by \( E_i \) and use that \( E_i E_j = 0 \)
for \( i \neq j \) \( \Rightarrow \ E_i^2 = E_i (E_1 + \ldots + E_k) = E_i I = E_i \).

(b) Let us assume only that \( E_1 + E_2 = I \), \( E_1^2 = E_1 \).
Then \( E_1 = E_1 I = E_1 (E_1 + E_2) = E_1^2 + E_1 E_2 = E_1 + E_1 E_2 \)
\( \Rightarrow E_1 E_2 = 0 \).
Similarly, \( E_1 = IE_1 = (E_1 + E_2) E_1 = E_1^2 + E_2 E_1 = E_1 + E_2 E_1 \)
\( \Rightarrow E_2 E_1 = 0 \).

and \( E_2 = IE_2 = (E_1 + E_2) E_2 = E_1 E_2 + E_2^2 = E_2 \)
\( \Rightarrow E_2^2 = E_2 \).