ANSWERS

1. a) \( u \cdot v = 3 \cdot 1 + (-1)0 + 2(-1) + 1(-1) = 0 \).
   
b) \( ||u|| = \sqrt{3^2 + (-1)^2 + 2^2 + 1^2} = \sqrt{15} \).

2. a) Note that

\[
\begin{align*}
  c_1 p_1 + c_2 p_2 + c_3 p_3 &= (c_1 + 2c_2 + c_3)x^2 + (-2c_1 + c_2 + 8c_3)x + (3c_1 + 8c_2 + 7c_3) \\
  &= (c_1 + 2c_2 + c_3)x^2 + (-2c_1 + c_2 + 8c_3)x + (3c_1 + 8c_2 + 7c_3)
\end{align*}
\]

is zero if and only if

\[
\begin{align*}
  c_1 + 2c_2 + c_3 &= 0 \\
  c_2 + 2c_3 &= 0.
\end{align*}
\]

Since this system has nontrivial solutions, the vectors \( p_1, p_2, p_3 \) are linearly dependent.

   b) If we set \( c_3 = -1 \), we get a solution of the above linear system with \( c_2 = 2 \) and \( c_1 = -3 \). Hence, \( p_3 = -3p_1 + 2p_2 \).

3. a) Gaussian elimination on \( A \) gives the row echelon matrix

\[
U = \begin{bmatrix}
  1 & 2 & 2 & 3 \\
  0 & 0 & 1 & 2 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}.
\]

A basis for \( \text{RS}(A) \) is the set of nonzero rows of \( U \), namely, \( \{(1, 2, 2, 3), (0, 0, 1, 2)\} \).
b) The pivots of $U$ are in columns 1 and 3. Then a basis for $\text{CS}(A)$ is given by the columns of $A$ in positions 1 and 3:

\[
\begin{bmatrix}
1 & 2 \\
2 & 5 \\
-1 & -3 \\
0 & 2
\end{bmatrix},
\begin{bmatrix}
1 \\
5 \\
-3 \\
2
\end{bmatrix}.
\]

c) $\text{NS}(A) = \text{NS}(U)$ is the set of solutions of the system $UX = 0$, which more explicitly is:

\[
x_1 + 2x_2 + 2x_3 + 3x_4 = 0 \\
x_3 + 2x_4 = 0.
\]

The leading variables are $x_1, x_3$ and the free variables are $x_2, x_4$. Set $x_2 = s, x_4 = t$ and solve for $x_1, x_3$ in terms of $s, t$. We obtain

\[
X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s + t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.
\]

A basis for $\text{NS}(A)$ is given by:

\[
\begin{bmatrix}
-2 & 1 \\
1 & 0 \\
0 & -2 \\
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}.
\]

d) $\text{rank}(A) = \text{dim \, RS}(A) = 2$.

4. a) We know that $u_1, u_2, u_3$ are linearly independent if and only if the matrix

\[
A = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}
\]

is invertible. This is true if and only if $A$ can be reduced by Gaussian elimination to an upper triangular matrix $U$ with nonzero diagonal
entries. In our case,

\[ U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \]

c) The coordinates of a column vector \( v \) with respect to a basis \( \{u_1, u_2, u_3\} \) are the unique scalars \( x_1, x_2, x_3 \) such that \( v = x_1 u_1 + x_2 u_2 + x_3 u_3 \). We find them by solving the linear system \( AX = v \) using Gaussian elimination on the augmented matrix

\[
[A \mid v] = [u_1 \quad u_2 \quad u_3] = \begin{bmatrix} 1 & 1 & 2 \mid 2 \\ 1 & 2 & 3 \mid 3 \\ 1 & 2 & 4 \mid 5 \end{bmatrix}.
\]

We find \( x_1 = x_2 = -1, \ x_3 = 2 \).