# Contents

*Preface and Acknowledgements*  

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><strong>The Essence of Probability</strong></td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>Trials, Sample Spaces, and Events</td>
<td>5</td>
</tr>
<tr>
<td>1.3</td>
<td>Probability Axioms and Relative Frequencies</td>
<td>15</td>
</tr>
<tr>
<td>1.4</td>
<td>Conditional Probability</td>
<td>23</td>
</tr>
<tr>
<td>1.5</td>
<td>Independent Events</td>
<td>29</td>
</tr>
<tr>
<td>1.6</td>
<td>Law of Total Probability</td>
<td>35</td>
</tr>
<tr>
<td>1.7</td>
<td>Bayes’ Rule</td>
<td>38</td>
</tr>
<tr>
<td>1.8</td>
<td>Simulating Probability Experiments</td>
<td>45</td>
</tr>
<tr>
<td>1.9</td>
<td>Exercises</td>
<td>51</td>
</tr>
<tr>
<td>2</td>
<td><strong>Combinatorics — The Art of Counting</strong></td>
<td>59</td>
</tr>
<tr>
<td>2.1</td>
<td>Permutations</td>
<td>60</td>
</tr>
<tr>
<td>2.2</td>
<td>Permutations with Replacements</td>
<td>61</td>
</tr>
<tr>
<td>2.3</td>
<td>Permutations without Replacement</td>
<td>65</td>
</tr>
<tr>
<td>2.4</td>
<td>Combinations without Replacement</td>
<td>68</td>
</tr>
<tr>
<td>2.5</td>
<td>Combinations with Replacements</td>
<td>77</td>
</tr>
<tr>
<td>2.6</td>
<td>Bernoulli (Independent) Trials</td>
<td>81</td>
</tr>
<tr>
<td>2.7</td>
<td>Exercises</td>
<td>86</td>
</tr>
<tr>
<td>3</td>
<td><strong>Random Variables and Distribution Functions</strong></td>
<td>91</td>
</tr>
<tr>
<td>3.1</td>
<td>Discrete and Continuous Random Variables</td>
<td>91</td>
</tr>
<tr>
<td>3.2</td>
<td>The Probability Mass Function for a Discrete RV</td>
<td>97</td>
</tr>
<tr>
<td>3.3</td>
<td>The Cumulative Distribution Function</td>
<td>104</td>
</tr>
<tr>
<td>3.4</td>
<td>The Probability Density Function for a Continuous RV</td>
<td>113</td>
</tr>
<tr>
<td>3.5</td>
<td>Functions of a Random Variable</td>
<td>117</td>
</tr>
<tr>
<td>3.6</td>
<td>Conditioned Random Variables</td>
<td>128</td>
</tr>
<tr>
<td>3.7</td>
<td>Exercises</td>
<td>136</td>
</tr>
</tbody>
</table>
4 Joint Random Variables and their Distributions 141
  4.1 Joint Probability Mass Functions ........................................ 141
  4.2 Joint Probability Density Functions ...................................... 154
     4.3.1 Uniformly Distributed Joint Random Variables ................ 160
     4.3.2 Deriving Joint CDFs from Joint Density Functions .......... 162
     4.3.3 Marginal Probability Density Functions and Independence .... 165
  4.3 Joint Cumulative Distribution Functions ............................... 148
  4.4 Conditional Distributions ................................................ 169
     4.4.1 Discrete Random Variables ....................................... 169
     4.4.2 Continuous Random Variables ................................... 172
  4.5 Convolutions and the Sum of Two Random Variables .................. 175
  4.6 Exercises ............................................................... 179

5 Expectations and More 183
  5.1 Definitions ........................................................................ 183
  5.2 Expectations of Functions of a Random Variable ..................... 192
  5.3 Jointly Distributed Random Variables ................................... 196
     5.3.1 Expectations and Variances ....................................... 196
     5.3.2 Covariances and Correlations ................................... 199
     5.3.3 Conditional Expectation and Variance ......................... 206
     5.3.4 Summary of Properties ............................................ 211
  5.4 Probability Generating Functions ........................................ 212
  5.5 Moment Generating Functions ............................................ 220
  5.6 Maxima and Minima of Independent Random Variables ............... 229
  5.7 Exercises ............................................................... 234

6 Discrete Probability Distributions 239
  6.1 The Discrete Uniform Distribution .................................... 239
  6.2 The Bernoulli Distribution .............................................. 242
  6.3 The Binomial Distribution ................................................ 244
  6.4 The Geometric Distribution ............................................. 250
  6.5 The Modified Geometric Distribution .................................. 254
  6.6 The Negative Binomial Distribution .................................... 255
  6.7 The Multinomial Distribution .......................................... 257
  6.8 The Hypergeometric Distribution ....................................... 260
  6.9 Poisson Distributions and Poisson Processes .......................... 264
     6.9.1 The Poisson Distribution ....................................... 264
     6.9.2 Relationship with the Binomial Distribution .................. 266
     6.9.3 The Poisson Process ............................................. 267
     6.9.4 Arrival and Inter-arrival Times .................................. 271
     6.9.5 The Distribution of Arrivals ................................... 273
     6.9.6 Superposition and Decomposition of Poisson Streams ........ 274
  6.10 Exercises ............................................................... 278
## CONTENTS

### 7 Continuous Probability Distributions 283
- 7.1 The Uniform Distribution ...................................................... 283
- 7.2 Wedge and Triangular Distributions ........................................... 287
- 7.3 The Exponential Distribution .................................................. 294
- 7.4 The Gamma Distribution ....................................................... 302
- 7.5 The Beta Distribution .......................................................... 310
- 7.6 The Weibull Distribution and Reliability Modeling ....................... 324
- 7.7 Phase-Type Distributions ....................................................... 335
  - 7.7.1 The Erlang Distribution .................................................... 336
  - 7.7.2 The Hypoexponential Distribution ....................................... 341
  - 7.7.3 The Hyperexponential Distribution ...................................... 343
  - 7.7.4 The Coxian Distribution ................................................... 346
  - 7.7.5 Fitting Phase-Type Distributions to Means and Variances .......... 350
- 7.8 Exercises .............................................................................. 358

### 8 Bounds and Limit Theorems 363
- 8.1 Three Probability Bounds ...................................................... 363
  - 8.1.1 The Markov Inequality .................................................... 363
  - 8.1.2 The Chebychev Inequality ................................................ 364
  - 8.1.3 The Chernoff Bound ........................................................ 365
- 8.2 The Laws of Large Numbers ................................................... 369
- 8.3 The Central Limit Theorem ..................................................... 372
- 8.4 Exercises .............................................................................. 377

### 9 Markov Chains 379
- 9.1 Discrete-Time Markov Chains: Definitions .................................. 380
- 9.2 The Chapman-Kolmogorov Equations ........................................ 389
- 9.3 Classification of States .......................................................... 395
- 9.4 Irreducibility .......................................................................... 404
- 9.5 Probability Distributions ........................................................ 408
- 9.6 Continuous-Time Markov Chains .............................................. 419
- 9.7 Exercises .............................................................................. 436

*Index* 442