Chapter 1

The Essence of Probability

1.1 Introduction

Probability is a branch of mathematics that affects our everyday lives to a much greater extent than we might otherwise have expected. When we say, “It looks like it will rain today, so I’d better take the umbrella”, we mean the probability that is will rain today is high so we had better be prepared, or “I think ‘fleet of foot’ has a good chance to win the Derby this year”, we believe the probability that our pick will win is better than 50-50, or when told that “the prognosis for Alice is not good”, we know that the probability that Alice will fully recover is low. Probabilities are everywhere around us. They are a major factor in determining the rates we pay for car insurance, for health insurance, for life insurance. Farmers must take weather-pattern probabilities into account when they make decisions as to when to plant their crops. Probability plays a fundamental role in our genetic makeup, in conducting census polling, in the number and gender of the children we have, and so on. And, of course, probabilities are omnipresent when we visit Las Vegas, play poker, try our hand at a slot machine, or appear on “Wheel of Fortune.”

Computing probabilities can sometimes be very deceptive and counter-intuitive. Consider the case of a participant in a game show who is shown three doors. He is told that behind one door is a brand new top-of-the-line automobile while behind each of the two other doors is a derisory prize, such as a rusty old bicycle (or a goat). Naturally, the participant would like to win the car. The game show host invites the player to choose one of the doors but forbids him to open it. At this point it seems reasonable to assume that the player has a 1 in 3 chance of choosing the winning door. The host then opens one of the other two doors and shows the player that it does not contain the car. The player is now asked if he wishes to keep the door he originally chose or switch to the other unopened door. It might be thought that since there are two doors left and the car is behind only one of them that it doesn’t really matter, that the probability that each contains the car is the same, so why bother switching? However, as we shall prove later in this chapter, the player actually doubles his probability of winning the
car by switching. This question was asked of Marilyn vos Savant, once listed in the
Guinness Book of World Records as the person with the highest IQ (228) in the world,
and of course, she gave the correct answer that the player should switch. A number
of mathematicians wrote the paper, some rather rudely, saying that she was incorrect
in her answer and that in future she should hire some mathematicians to work along
with her. These mathematicians later had to acknowledge their own mistake and apol-
gogize. We shall return to this problem later, but it serves to show that working with
probabilities can be hazardous to a person’s well-being.

Another widely cited probability problem whose solution may be considered counter-
intuitive is the “Birthday problem.” If we take a year to consist of 365 days, thereby
avoiding any complicating issues with leap-years, then in a group of 23 people, there
is a better than 50-50 chance that at least two of them have the same birthday (day
and month only). Given that 23 is relatively small compared with 365, this may seem
surprising, but nevertheless it is true. In a group of 30 people, the probability that at
least two have a common birthday rises to 71%, in a group of 40 it is 89%, while in a
group consisting of only 50 people, the probability that at least two share a birthday is
97%.

It is therefore important to approach the study of probability with an open mind
and without prejudice of any kind. To correctly solve probability problems it is imper-
ative that all the underlying assumptions be fully understood and that all ambiguity be
removed from the problem statement. When this is the case, then the comment of the
famous French mathematician Pierre-Simon Laplace (1749–1827) is apropos: “Prob-
ability theory is nothing but common sense reduced to calculation.” We also note in
passing that Laplace is also reported to have said, “Life’s most important questions are,
for the most part, nothing but probability problems.”

It is widely acknowledged that the study of probability has its origins in gambling and
although both gambling and mathematics existed side by side for thousands of years, it
was not until the 1600s that gambling first underwent a rigorous mathematical analysis.
It arose out of questions posed by a member of the French nobility and avid gambler,
Antoine Gombaud (1607–1648), more commonly known as Chevalier de Mérè, to his
friend and mathematician Blaise Pascal (1623–1662). Pascal in turn communicated
with another brilliant French mathematician Pierre de Fermat (1601–1655) and their
surviving letters detail what is now recognized as the birth of probability theory. We
now briefly describe two problems posed by de Mérè and analyzed by Pascal and Fermat.

If a fair six-side die is tossed once, the probability of obtaining a six is 1/6; the prob-
ability of getting something other than a six is 5/6. In the time of de Mérè (and indeed
still today, as we shall prove later in this chapter) it was known that the probability of
getting at least one six in four tosses is

\[
\text{Prob}\{\text{At least one six in 4 tries}\} = 1 - \text{Prob}\{\text{Not a single six in 4 tries}\} = 1 - \left(\frac{5}{6}\right)^4 \approx .52
\]
so that someone betting on getting at least one six is slightly favored. However, this was not the game that was generally played. Instead two dice were tossed and the gambler wins if at least a pair of sixes appear in 24 tosses. Since there are six times as many possible results with two dice than with one, it was assumed that six times as many tosses, i.e., 24, would be needed to maintain the gambler’s edge. De Mérè became suspicious of this, perhaps due to his numerous losses, and hence his turn to Pascal. The French mathematicians were able to prove that the probability of winning is now:

\[ \text{Prob}\{\text{At least one pair of sixes in 24 tries}\} = 1 - \left( \frac{35}{36} \right)^{24} \approx .49 \]

so now the gambler no longer has an edge. Instead the advantage is with the house, and in the long run the gambler will lose all his money.

De Mérè also consulted his mathematician friend on another well-known gambling problem. Suppose two friends compete on an equal footing, tossing an unbiased coin, for example. With each toss, one of the friends wins when a head appears while the other wins when a tail appears. Thus each has probability of one-half to win or lose on each toss. Each player puts in $50 at the start of the game and whomever gets to seven wins first takes all $100. Suppose that when the score is 6-5 the game comes to an abrupt end. How should the $100 be split between the two players? This situation is referred as the “problem of points.” Should each take his $50 back? Should the first player having already won six times get 6/11 of the pot and his friend 5/11? According to the French mathematicians neither of these is fair. Instead, they concluded that the fair resolution is that each player should get an amount in proportion to their chances of winning the game. The player with only five wins would need to win both of the next two consecutive tosses, and having a probability of one-half of winning each, has only a one in four chance of taking the $100; and hence the player with six wins has a three in four chance of winning the game. Thus, the player with six wins should be given $75 and the one with five wins, $25. Communications between Pascal and Fermat on these problems are now widely viewed as the foundations of the systematic study of probability.

We terminate this introduction by returning to the opening paragraphs and providing a solution to the game-show problem. As we mentioned at that time, it is important that all assumptions be clear and understood by all. In this particular problem, the implied assumption is that the game show host knows which door contains the prize (and hence which do not). If this were not the case, then, from time to time as different participants come and go, the host would open the door containing the car and in the context of the game, that would be a pointless conclusion.

It is enlightening to suppose that instead of just 3 doors, there are 100 doors and that the prize is randomly located behind just one of them. If you were to choose one of these 100 doors, then logically there is a 1 in 100 chance that you were lucky enough to choose correctly; conversely there is a 99 in 100 chance that the prize is behind one of the other doors. The game show host is now going to tell you which of those other 99 doors has that 99 in 100 chance of holding the prize — by eliminating 98 of them.