Practice Final Solutions

Show all work.

1. (8 points) Use the method of least squares to find the straight line that best fits the points \((2, 2), \ (3, 0), \ (7, -1)\).

Solution: Solution 1:

\[
E(A, B) = (2A + B - 2)^2 + (3A + B)^2 + (7A + B + 1)^2.
\]

Then

\[
\frac{\partial E}{\partial A} = 2 \cdot (2A + B - 2) \cdot 2 + 2 \cdot (3A + B) \cdot 3 + 2 \cdot (7A + B + 1) \cdot 7 = 124A + 24B + 6 = 0
\]

\[
\frac{\partial E}{\partial B} = 2 \cdot (2A + B - 2) + 2 \cdot (3A + B) + 2 \cdot (7A + B + 1) = 24A + 6B - 2 = 0.
\]

This gives the solution \(A = -1/2\) and \(B = 7/3\). The equation of the line is \(y = -\frac{1}{2}x + \frac{7}{3}\).

Solution 2:

From the table

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(x^2)</th>
<th>(xy)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>-1</td>
<td>49</td>
<td>-7</td>
</tr>
<tr>
<td>(\sum)</td>
<td>12</td>
<td>1</td>
<td>62</td>
</tr>
</tbody>
</table>

we get that

\[
A = \frac{(3)(-3) - (12)(1)}{(3)(62) - (12)^2} = \frac{-21}{42} = -\frac{1}{2} \quad B = \frac{(1) - \frac{7}{3}(12)}{3} = \frac{7}{3}.
\]

The equation of the line is \(y = -\frac{1}{2}x + \frac{7}{3}\).

2. (10 points) Evaluate

\[
\int_{R} (x^2 y + y^2) \, dx \, dy
\]

where \(R\) is the rectangle \(0 \leq x \leq 2\) and \(2 \leq y \leq 3\).

Solution:

\[
\int_{2}^{3} \int_{0}^{2} (x^2 y + y^2) \, dx \, dy = \int_{2}^{3} \left[ \frac{x^3}{3} y - \frac{y^3}{3} x \right]_{x=0}^{x=2} \, dy = \frac{2}{3} \int_{2}^{3} (45 - 16) = \frac{58}{3}.
\]
3. (12 points) Find the critical points of
\[ f(x, y) = x^3 - y^2 - 3x + 4y, \]
and determine whether they are minima, maxima or saddle points.

Solution:
\[
\frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \Rightarrow x = 1 \text{ or } x = -1
\]
\[
\frac{\partial f}{\partial y} = -2y + 4 = 0 \Rightarrow y = 2,
\]
so the critical points are (1, 2) and (-1, 2). Since
\[
\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0,
\]
we have that
\[
D(x, y) = 6x \cdot (-2) - (0)^2 = -12x,
\]
therefore
\[
D(1, 2) = -12 < 0 \Rightarrow (1, 2) \text{ is a saddle point}
\]
\[
D(-1, 2) = 12 > 0, \text{ and } \frac{\partial^2 f}{\partial x^2}(-1, 2) = -6 < 0 \Rightarrow (-1, 2) \text{ is a local maximum}
\]

4. (10 points) (a) Sum the series
\[
\frac{3}{4} - \frac{3}{16} + \frac{3}{64} - \frac{3}{256} + \ldots
\]
(b) A doctor wishes to give a patient a daily dose of \( D \) mgs of a certain medicine. The doctor knows that the body eliminates each day 70% of the amount present in the body. In the long run, the doctor would like the amount of drug in the body to approach 6 mgs. What should \( D \) be?

Solution:

(a)
\[
\frac{3}{4} - \frac{3}{16} + \frac{3}{64} - \frac{3}{256} + \ldots = \frac{3}{4} \left( \frac{1}{1 + \frac{1}{4}} \right) = \frac{3}{4} \cdot \frac{1}{5} = \frac{3}{5}.
\]

(b)
\[
D + 0.3D + (0.3)^2D + \ldots = D \left( \frac{1}{1 - 0.3} \right) = \frac{D}{0.7} = 6 \Rightarrow D = 4.2.
\]

5. (8 points) Find the third Taylor polynomial at \( x = 1 \) for the function \( f(x) = 1/(x+1) = (x+1)^{-1} \).

Solution:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{[n]}(x) )</th>
<th>( f^{[n]}(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( (x+1)^{-1} )</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( -(x+1)^{-2} )</td>
<td>( -\frac{1}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>( 2(x+1)^{-3} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>3</td>
<td>( -6(x+1)^{-4} )</td>
<td>( -\frac{3}{2} )</td>
</tr>
</tbody>
</table>
\[
P_3(x) = \frac{1}{2} - \frac{1}{4}(x - 1) + \frac{1}{4 \cdot 2!}(x - 1)^2 - \frac{3}{8 \cdot 3!}(x - 1)^3
\]
\[
= \frac{1}{2} - \frac{1}{4}(x - 1) + \frac{1}{8}(x - 1)^2 - \frac{3}{48}(x - 1)^3.
\]

6. **(12 points)** Use separation of variables to solve the following differential equations:

(a) \(\frac{dy}{dt} = yt, \ y(0) = 1\);  \(b) \frac{dy}{dt} = y^3t + y^3, \ y(0) = 1\).

**Solution:**

(a)

\[
\frac{dy}{dt} = yt
\]
\[
\int \frac{1}{y} dy = \int t dt
\]
\[
\ln y = \frac{t^2}{2} + C
\]
\[
y = Ae^{\frac{t^2}{2}}
\]
\[
y(0) = 1 \Rightarrow A = 1
\]
\[
y(t) = e^{\frac{t^2}{2}}.
\]

(b)

\[
\frac{dy}{dt} = y^3t + y^3
\]
\[
\int \frac{1}{y^2} dy = \int (t + 1) dt
\]
\[
\frac{1}{2y^2} = \frac{t^2}{2} + t + C
\]
\[
y = \frac{1}{-t^2 - 2t - C}
\]
\[
y(0) = 1 \Rightarrow C = -1
\]
\[
y(t) = \frac{1}{-t^2 - 2t + 1}.
\]

7. **(8 points)** If a radioactive substance decays from 5 grams to 3 grams in 6 days, when will only 1 gram remain?

**Solution:**

The general formula for \(y(t)\), the amount of the radioactive material left after \(t\) days, is

\[
y(t) = Ce^{-kt}.
\]
We can determine the value of the constants $C$ and $k$ from the initial conditions as follows:

$$y(0) = Ce^{-0k} = C = 5$$
$$y(6) = 5e^{-6k} = 3 \Rightarrow e^{-6k} = 3/5 \Rightarrow k = \frac{\ln(3/5)}{-6} \approx 0.085,$$

thus the formula for $y$ is

$$y(t) = 5e^{-0.085t}.$$ 

Next we need to find $t$ such that $y(t) = 1$. Using the formula above, we get

$$5e^{-0.085t} = 1 \Rightarrow e^{-0.085t} = 1/5 \Rightarrow -0.085t = \ln(1/5) \Rightarrow t = \frac{\ln(1/5)}{-0.085} \approx 18.94.$$

8. **(10 points)** For the differential equation $dy/dt = (y+1)(y-7)$ sketch the constant solutions and the solutions corresponding to $y(0) = -3$, $y(0) = 2$, $y(0) = 10$.

**Solution:**
Constant solutions occur when the right hand side of the equation is zero:

$$(y + 1)(y - 7) = 0 \Rightarrow y = -1 \text{ or } y = 7.$$ 

The horizontal lines $y = -1$ and $y = 7$ cut the $t$-$y$ plane into 3 regions. Next we compute the slopes of the tangents at $t = 0$ of the solutions corresponding to the three initial conditions:

$$y(0) = -3 \Rightarrow \frac{dy}{dt}(0) = (y(0) + 1)(y(0) - 7) = (-3 + 1)(-3 - 7) = 20 > 0$$
$$y(0) = 2 \Rightarrow \frac{dy}{dt}(0) = (y(0) + 1)(y(0) - 7) = (2 + 1)(2 - 7) = -15 < 0$$
$$y(0) = 10 \Rightarrow \frac{dy}{dt}(0) = (y(0) + 1)(y(0) - 7) = (10 + 1)(10 - 7) = 33 > 0$$

This implies that the solutions in the bottom region are increasing, the solutions in the middle region are decreasing, and the solutions in the top region are increasing. The sketch of the solutions are as follows:

<table>
<thead>
<tr>
<th>10</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>-4</td>
<td>-1</td>
</tr>
</tbody>
</table>

9. **(10 points)** A scientist is growing a bacteria culture. She knows that the bacteria grow at a rate of 3% per day. In addition, she wants to remove $M$ bacteria
each day. Initially she has a culture of 500,000 bacteria.

(a) Find a differential equation for $y(t)$, the population size after $t$ days.

(b) What is the maximum number of bacteria she can remove each day without eventually killing off the population of bacteria?

Solution:

(a) \[
\frac{dy}{dt} = 0.03y - M
\]
\[
y(0) = 500,000
\]

(b) If \(\frac{dy}{dt}\) starts negative then the population will eventually die out. So \(\frac{dy}{dt}(0)\) needs to be \(\geq 0\).

\[
\frac{dy}{dt}(0) \geq 0 \Rightarrow 0.03y(0) - M \geq 0 \Rightarrow M \leq 0.03y(0) \Rightarrow M \leq 0.03 \times 500,000 = 15,000.
\]

10. (6 points) Use two iterations of the Newton-Raphson method to approximate a zero of \(x^3 - x - 2\) between 1 and 2. Let \(x_0 = 1\).

Solution:
\[
f(x) = x^3 - x - 2, \quad f'(x) = 3x^2 - 1
\]
Iteration 0: \(x_0 = 1\), \(f(1) = -2\), \(f'(1) = 2\).
Iteration 1: \(x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-2}{2} = 1 + 1 = 2\), \(f(2) = 4\), \(f'(2) = 11\).
Iteration 2: \(x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{4}{11} = 1.64\).

11. (6 points) Use Euler’s method with \(n = 2\) to approximate \(y(2)\) where \(y(t)\) is a solution of \(\frac{dy}{dt} = t - 2y\) and \(y(0) = 2\).

Solution:
\[
\frac{dy}{dt} = t - 2y, \quad y(0) = 2, \quad n = 2, a = 0, b = 2, \quad h = \frac{b - a}{n} = \frac{2}{2} = 1.
\]
\[
y(0) = 2
\]
\[
y(1) \cong y(0) + h(t - 2y(0)) = 2 + 1(0 - 2 \cdot 2) = -2
\]
\[
y(2) \cong y(1) + h(t - 2y(1)) = -2 + 1(1 - 2 \cdot (-2)) = 3
\]