Solutions to Practice Problem Set 3

1(a). This is immediate upon noting that for fixed \( n \), \( \lim_{m \to \infty} S_{m,n} = 1 \).

1(b). This is immediate upon noting that for fixed \( m \), \( \lim_{n \to \infty} S_{m,n} = 0 \).

2. Note that \( f : \mathbb{R} \to [0, \infty) \). However, for the purposes of this answer, we confine attention to \( x \in \mathbb{R}^+ \). Over this part of the domain, if

\[
\frac{x^{2T}}{1 + x^{2T}} = a
\]

then there is a unique inverse function

\[
x = \exp \left\{ \left( \frac{1}{2T} \right) \ln \left[ \frac{a}{1-a} \right] \right\}
\]

(2)

We show that for any \( T = T_1 \), say, we can always find a \( T_2 \) such that the Cauchy condition is violated for some \( x \). To this end, note that for any \( T_1 \), we can find \( x \) such that \( f_{T_1}(x) = 0.5 + \eta \) for some arbitrarily small positive constant. Let this value of \( x \) be denoted \( \tilde{x} \). From (2), it follows that if \( \tilde{x} > 1 \) then \( \tilde{x} > 1 \) and \( \tilde{x} = \exp \left\{ \left( \frac{1}{2T_1} \right) \ln \left[ \frac{0.5}{0.5-0.5+\eta} \right] \right\} \) is not equicontinuous.

3. Following the answer to Question 2, we define \( \tilde{x} \) and \( T \) such that \( 1 < \tilde{x} < 1 + \delta \) for \( \delta > 0 \) and \( f_{T}(\tilde{x}) = 1 - \eta \) for arbitrarily small \( \eta > 0 \). From (2) it follows that there exists \( \tilde{x} \) such that \( 1 < \tilde{x} < \tilde{x} < 1 + \delta \) such that \( f_{T}(\tilde{x}) = 0.5 + \eta \). Therefore, \( ||\tilde{x} - \tilde{x}|| < \delta \) and \( ||f_{T}(\tilde{x}) - f_{T}(\tilde{x})|| = 0.5 - 2\eta > \epsilon \) for some \( \epsilon > 0 \). Since the previous statements hold for any \( \delta \), \( \{f_{T}\} \) is not equicontinuous.

4. Assumption 3.2 (i) states that \( f(v_t, \theta) \) exists for all \( \theta \in \Theta \) and each \( v_t \in \mathcal{V} \). Assumption 3.5 (i) states that \( \partial f(v_t, \theta)/\partial \theta \) exists for all \( \theta \in \Theta \) and all \( v_t \in \mathcal{V} \). Therefore,

\[
\frac{\partial Q_T(\theta)}{\partial \theta} = 2 \left\{ T^{-1} \sum_{t=1}^{T} \frac{\partial f(v_t, \theta)}{\partial \theta} \right\} W_T T^{-1} \sum_{t=1}^{T} f(v_t, \theta)
\]

exists for all \( \theta \in \Theta \). Using the Mean Value Theorem, it follows that

\[
Q_T(\tilde{\theta}) - Q_T(\hat{\theta}) = \sum_{i=1}^{q} \frac{\partial Q_T(\theta_i)}{\partial \theta_i} (\hat{\theta}_i - \theta_i)
\]

(3)

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where $\theta^* = \lambda \tilde{\theta} + (1 - \lambda)\theta$ for some $\lambda \in [0, 1]$. It follows from (3) that

$$|Q_T(\tilde{\theta}) - Q_T(\tilde{\theta})| = \left| \sum_{i=1}^{q} \frac{\partial Q_T(\theta)}{\partial \theta_i} (\tilde{\theta}_i - \theta_i) \right|$$

$$\leq \sum_{i=1}^{q} \left| \frac{\partial Q_T(\theta^*)}{\partial \theta_i} \right| |(\tilde{\theta}_i - \theta_i)|$$

Using the Cauchy-Schwarz inequality ($\sum_{i=1}^{m} a_i b_i \leq \sqrt{\sum_{i=1}^{m} a_i^2} \sqrt{\sum_{i=1}^{m} b_i^2}$), it follows from (5) that

$$|Q_T(\tilde{\theta}) - Q_T(\tilde{\theta})| \leq \left\| \frac{\partial Q_T(\theta^*)}{\partial \theta} \right\| \left\| \tilde{\theta} - \theta \right\|$$

$$\leq \sup_{\theta \in \Theta} \left\| \frac{\partial Q_T(\theta)}{\partial \theta_i} \right\| \left\| \tilde{\theta}_i - \theta_i \right\|$$

From (7), it can be seen that if $\sup_{\theta \in \Theta} \left\| \frac{\partial Q_T(\theta)}{\partial \theta} \right\| = O_T(1)$ then the Lipschitz condition is satisfied with $B_T = \sup_{\theta \in \Theta} \left\| \frac{\partial Q_T(\theta)}{\partial \theta} \right\|$, and $h(c) = c$. 

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