**Practice Problem Set 4**

1. In class, we introduced the norm of a $m \times n$ matrix $A$: $\| A \| = \sqrt{tr(\nabla^2 A)}$; this is known as the Frobenius norm. Show that $\| A \| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j}^2}$ where $a_{i,j}$ is the $i \times j$ element of $A$.

2. Show that:
   (a) if $m = n = 1$ then $\| A \| = |A|$;
   (b) if $m > 1$ and $n = 1$ or $m = 1$ and $n > 1$ then $\| A \|$ equals the Euclidean norm for vector $A$;
   (c) $\| A \|$ equals the Euclidean norm of $vec(A)$.

3. Three basic properties of a matrix norm, $f(A)$, are:
   (a) $f(A) \geq 0$ for all $A \in \mathbb{R}^{m \times n}$ and $f(A) = 0$ iff $A = 0^{m \times n}$;
   (b) (The Triangle Inequality) $f(A + B) \leq f(A) + f(B)$ for all $A, B \in \mathbb{R}^{m \times n}$;
   (c) $f(\alpha A) = |\alpha|f(A)$ for $\alpha \in \mathbb{R}$, $A \in \mathbb{R}^{m \times n}$.

   Show that the Frobenius norm satisfies these three conditions.

4. (Cauchy-Schwarz inequality for matrices) Show that $\| AB \| \leq \| A \| \| B \|$.

5. Show that $\sup_{i,j}|a_{i,j}| \leq \| A \| \leq \sqrt{mn} \sup_{i,j}|a_{i,j}|$.

6. Let $A_T$ be a sequence of random $m \times n$ matrices with $i \times j$ element $a_{T,i,j}$. Show that $\| A_T - A \| \xrightarrow{P} 0$ iff $a_{T,i,j} \xrightarrow{P} a_{i,j}$ for all $i,j$ where $a_{i,j}$ is the $i \times j$ element of $A$.

7. In this question, we establish the uniform convergence of the GMM minimand under the conditions stated in Lemma 3.1 in the text. To this end, we first present the Uniform Law of Large Numbers (ULLN).

   **Uniform Law of Large Numbers:** If Assumptions 3.1, 3.2, 3.8, 3.9 and 3.10 hold then: $\sup_{\theta \in \Theta} \| g_T(\theta) - E[f(v, \theta)] \| = o_p(1)$.

   (a) Let $a(\theta) = E[f(v, \theta)]$. Show that

   $$Q_T(\theta) - Q_0(\theta) = g_T(\theta)'W_Tg_T(\theta) - a(\theta)'Wa(\theta)$$

   $$= \{g_T(\theta) - a(\theta)\}'W_T\{g_T(\theta) - a(\theta)\} + a(\theta)'\{W_T - W\}a(\theta) + 2a(\theta)'W_T\{g_T(\theta) - a(\theta)\}$$
(b) Using part (a), show that

\[ |Q_T(\theta) - Q_0(\theta)| \leq \| g_T(\theta) - a(\theta) \|^2 \| W_T \| + \| a(\theta) \|^2 \| W_T - W \| + 2 \| a(\theta) \| \| W_T \| \| g_T(\theta) - a(\theta) \| \]

(c) Using part (b), show \( \sup_{\theta \in \Theta} |Q_T(\theta) - Q_0(\theta)| = o_p(1) \) under the conditions stated in Lemma 3.1.