Practice Problem Set 2

1. Show that $E[|v|^k] < \infty$ for positive $k$ implies $E[|v|^j] < \infty$ for $0 < j < k$.

2. Jensen’s inequality states: If $g(.)$ is a convex function, $E[|g(v)|] < \infty$ and $E[|v|] < \infty$ then $g(E[|v|]) \leq E[g(v)]$. Assuming that $E[|v|^k] < \infty$, show that \( \{E[|v|^j]\}^{1/j} \leq \{E[|v|^k]\}^{1/k} \) for $j < k$. Hint: $|\cdot|^c$ is a convex function for all $c > 1$.

3. Show that $E[|v|^j] < \infty$ implies $E[|v|^j] < \infty$.

4. Show that the characteristic function always exists, that is for any random variable $v$, $E[e^{in\omega}] < \infty$. Hint: $e^{ia} = \cos(a) + i\sin(a)$ and $|a + ib| = (a^2 + b^2)^{1/2}$.

5. Show that the characteristic function of the standard normal distribution is: $c(n) = e^{-n^2/2}$

6. Use the characteristic function in Question 4 to derive the first four moments of the standard normal distribution.

7. Consider the univariate AR(1) process

\[ u_t = \theta u_{t-1} + w_t \]

where $w_t \sim iid(0, \sigma_w^2)$ and $|\theta| < 1$. It can be shown that the $j^{th}$ autocovariance of $u_t$ is equal to

\[ \gamma_j = \frac{\sigma_w^2 \theta^j}{1 - \theta^2}, \quad \text{for } j = 0, 1, 2, \ldots \]

Show that the autocovariances of $u_t$ are absolutely summable.