SAS Handout # 6

Estimation of ARCH models

In this handout we consider the estimation of ARCH models for the US dollar trade weighted exchange index. The data are weekly and span the period 1/13/71 - 1/6/1999. The data and program have been mailed to your computer account.

To begin, it is useful to plot the time path of the natural logarithm of the exchange rate index.

![Time series plot of log index](image)

Figure 1: Time series plots of the log index

Although there is no obvious trend in the data, the series does not look stationary. In fact, it looks more like a random walk. Therefore, we now consider the first differences of the log of the exchange rate index. As can be seen from Figure 2, the first difference look like a stationary process.
Two other features stand out. First, the series fluctuates around zero and there is little evidence to suggest that the conditional mean would be linearly predictable (a common finding in financial data that is predicted by the Efficient Markets hypothesis). Second, the series has periods of low volatility and periods of high volatility. This would suggest that the conditional variance changes over time.

ARCH models can be estimated in SAS using proc autoreg. Since the plots suggest that the conditional mean is linearly unpredictable but the conditional variance is time varying, we begin with the ARCH(1) model

\[
\begin{align*}
    v_t &= \mu_0 + h_t^{1/2} z_t \\
    h_t &= \zeta_0 + \alpha_1 (v_{t-1} - \mu_0)^2 \\
    z_t &\sim \text{IN}(0, 1)
\end{align*}
\]

This is estimated in SAS as follows.

```sas
proc autoreg data=exchange;
model v= / garch=(q=1);
output out=a cev=v
```

These commands play the following roles:

- **proc autoreg data=exchange**: invokes the autoreg procedure and tells SAS to use the data set exchange;
model v=/ garch=(q=1): tells SAS to estimate a model for the series v that contains only the intercept in the mean and uses an ARCH(1) model for the conditional variance;

output out=a cev=v: tells SAS to print the estimated conditional variances into a data set called a and to call these estimated variances v therein.

These commands cause the parameters to be estimated via conditional ML. The output is given in the appendix. The results suggest that ARCH effects are present. An impression reinforced by examining the time plot of the estimated ARCH(1) model given in Figure 3.

Figure 3: Estimated conditional variance from ARCH(1) model

However, the choice of lag is arbitrary and we would naturally wonder if the model adequately captures the time series behaviour of the conditional variance. As with fitting models for the conditional mean, a natural approach to lag length selection in ARCH models is to include a large number of lags and then test their significance in some fashion. So we next consider the ARCH(q) model

\[ v_t = \mu_0 + h_t^{1/2} z_t \]

\[ h_t = \zeta_0 + \sum_{i=1}^{q} \alpha_{0,i} (v_{t-i} - \mu_0)^2 \]

\[ z_t \sim IN(0,1) \]

Some experimentation reveals that an ARCH(1) model is indeed too parsimonious. To illustrate, we consider the results for an ARCH(15) model. When this estimation is performed, it turns out
that the numerical optimization does not converge within the default maximum number of iterations (which is 50). Therefore, this ceiling needs to be increased. For example, it can be set at 100 for the ARCH(15) model as follows.\footnote{It is also worth noting that \texttt{autoreg} has a default convergence criterion of 0.001. This can be altered to 0.0005, say, by including the option \texttt{convergence=0.0005} in the model statement.}

\begin{verbatim}
proc autoreg data=exchange;
model v=/ garch=(q=15) maxiter=100;
output out=garch=(q=15) maxiter=100;
\end{verbatim}

Although not all the coefficients are significant, the additional fourteen parameters are collectively significant. For example, the LR statistic for $H_0 : \alpha_{0,i} = 0, i = 2 \ldots 15$ is $LR = 2(4883.07 - 4755.87) = 334.4$ which has a p-value of zero effectively. The inclusion of these additional terms also affects the estimated conditional variances. A comparison of Figures 3 and 4 indicates the estimated conditional variances are different for the two models, and particularly around 1985.

![Figure 4: Estimated conditional variances from ARCH(15) model](image)

One problem with an ARCH(15) model is that it requires a relatively large number of parameters to capture the conditional variance structure. However, it may be possible to capture this structure using far fewer parameters by using what is known as \textit{Generalized ARCH} - or GARCH - model. To motivate the GARCH model, suppose that $h_t$ has the following infinite order ARCH representation:

$$h_t = \zeta + A(L)\nu_t^2$$
where \( A(L) = \sum_{i=1}^{\infty} \alpha_i L^i \). Using the same argument as we used to move from the Wold decomposition to ARMA models, we assume that

\[
A(L) = \frac{B(L)}{1 - D(L)}
\]

where \( B(L) = \sum_{i=1}^{q} \beta_i L^i \) and \( D(L) = \sum_{i=1}^{p} \delta_i L^i \) and \((p, q)\) are finite. Subject to the usual restrictions on the roots of the polynomial in the denominator, we thus obtain

\[
h_t = \nu + D(L) h_t + B(L) v_t^2
\]  

where \( \nu = (1 - D(1)) \zeta \). The GARCH(p,q) model is:

\[
\begin{align*}
\nu_t &= \mu_0 + h_{t-1}^{1/2} z_t \\
h_t &= \nu + \sum_{i=1}^{p} \delta_i h_{t-i} + \sum_{i=1}^{q} \alpha_i (v_{t-i} - \mu_0)^2 \\
z_t &\sim IN(0,1)
\end{align*}
\]

We first consider a GARCH(1,1) model. This requires only a minor modification of the commands for the ARCH model to give:

```plaintext
proc autoreg data=exchange;
model v=/ garch=(p=1,q=1);
output out=a cev=v
```

The output suggests that all the coefficients are significant. It is interesting to compare the estimated conditional variances with their ARCH counterparts. A comparison of Figures 3, 4 and 5 indicates that the GARCH(1,1) is more like the ARCH(15) model than the ARCH(1) in the sense that it yields relatively large estimated conditional variances around 1985. However, the ARCH(15) yields higher estimates than the GARCH(1,1) around 1985.
Figure 5: Estimated conditional variances from GARCH(1,1) model

Once again, the lag lengths have been arbitrarily chosen. Some experimentation suggest that the GARCH(3,2) may be appropriate. The output for this model is given in the output appendix. The estimated conditional variances are given in Figure 6.
A comparison of Figures 4 and 6 shows that the estimated conditional variances are very similar for the ARCH(15) and GARCH(3,2) model. The difference is that the conditional variance involves sixteen parameters for the ARCH(15) model and only six for the GARCH(3,2).