Practice Problem Set 3

1. Let $v_t$ be generated via the $AR(p)$ model

$$\theta(L)v_t = w_t$$

where $\theta(L) = 1 - \theta_1 L - \theta_2 L^2 - \ldots - \theta_p L^p$. Show that (1) can be rewritten as the following VAR(1) process

$$x_t = Fx_{t-1} + \epsilon_t$$

where $x_t = [v_t, v_{t-1}, v_{t-2}, \ldots, v_{t-p+1}]'$, $\epsilon_t = [w_t, 0, 0, \ldots, 0]'$ and

$$F = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \ldots & \theta_{p-1} & \theta_p \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

2. Let $\{\lambda_i; i = 1, 2\}$ be the eigenvalues of

$$\Theta = \begin{bmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{bmatrix}$$

and assume that $\{\lambda_i\}$ are real, distinct (i.e. $\lambda_1 \neq \lambda_2$) and less than one in absolute value.

(a) Show that $I - \Theta$ is nonsingular.

(b) Show that

$$(I_2 - \Theta)^{-1} = I_2 + \sum_{i=1}^{\infty} \Theta^i$$

3. Let $v_t$ be the $2 \times 1$ vector generated by the VAR(1) process

$$v_t = \Theta v_{t-1} + w_t$$

where $w_t \sim i.i.d.(0, \Omega)$ and

$$\Theta = \begin{bmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{bmatrix}$$

From class, we know that the condition for the stationarity of $v_t$ is that the roots of $|I_2 - \Theta m| = 0$ are outside the unit circle. Assume that $\theta_{i,j} = 0$ for $i \neq j$ and show this stationarity condition is satisfied if $|\theta_{i,i}| < 1$ for $i = 1, 2$. 

1
4. Suppose that \( v_t \) is a \( k \times 1 \) process with \( E[v_t] = \mu \), and that \( v_t - \mu \) is a stationary VAR(p) process with AR polynomial \( \Theta(L) = I_k - \sum_{i=1}^{p} \Theta_i L^i \) and error term \( w_t \). Show that \( v_t \) has the following representation

\[
v_t = \nu + \Theta_1 v_{t-1} + \Theta_2 v_{t-2} + \ldots + \Theta_p v_{t-p} + w_t
\]

where \( \nu = \Theta(1)\mu \).

5. Let \( v_t \) be the \( k \times 1 \) vector generated by the stationary VAR(1) process

\[
v_t = \Theta v_{t-1} + w_t
\]

where \( w_t \sim i.i.d. (0, \Omega) \).

(a) Show that \( \text{Var}[v_t] = \Omega + \sum_{i=1}^{\infty} \Theta^i \Omega \Theta^i \).

(b) Show that \( \text{Cov}[v_t, v_{t-j}] = \Theta^j \text{Var}[v_t] \).

(c) Verify that for \( k = 1 \) the results in (a)-(b) correspond with those derived earlier in class for a univariate AR(1) process.