Practice Problem Set 5

1. Consider the ARCH(1) model

\[
\begin{align*}
    w_t &= h_t^{1/2} z_t \\
    h_t &= \zeta + \alpha w_{t-1}^2 \\
    z_t &\sim \text{IN}(0,1)
\end{align*}
\]

Assume that \( w_t \) is fourth order stationary so that \( E[w_t^i w_t^j w_t^k w_t^\ell] \) is finite and independent of \( (a, b, c, d) \) for all \( i + j + k + \ell \leq 4 \) for nonnegative integers \( i, j, k, \ell \). Define \( V_t = (w_t, w_{t-1}, w_{t-2}, \ldots) \).

Show: (a) \( E[w_t|V_{t-1}] = 0 \), \( \text{Var}[w_t|V_{t-1}] = h_t \), \( E[w_t^3|V_{t-1}] = 0 \), \( E[w_t^4|V_{t-1}] = 3\zeta^2(1 + \alpha)/(1 - \alpha)(1 - 3\alpha^2) \), \( E[w_t w_s|V_{t-1}] = 0 \) for \( t > s \); (b) \( E[w_t] = 0 \), \( \text{Var}[w_t] = \zeta/(1 - \alpha) \), \( E[w_t^3] = 0 \), \( E[w_t^4] = 3\zeta^2(1 + \alpha)/(1 - \alpha)(1 - 3\alpha^2) \), \( E[w_t w_s] = 0 \).

2. Consider the AR(1)-ARCH(1) model

\[
\begin{align*}
    v_t &= c + \theta v_{t-1} + h_t^{1/2} z_t \\
    h_t &= \zeta + \alpha w_{t-1}^2 \\
    w_t &= h_t^{1/2} z_t \\
    z_t &\sim \text{IN}(0,1)
\end{align*}
\]

Define \( \beta = (\eta', \lambda')' \) for \( \eta = (c, \theta)' \) and \( \lambda = (\zeta, \alpha)' \). Consider the score equations associated with the conditional likelihood based on sample \( \{v_t; t = 1, 2 \ldots T\} \) given \( u_0, v_{-1} \) that is,

\[
\sum_{t=1}^{T} s_t(\hat{\beta}_T) = 0
\]

Show that

\[
s_t(\beta) = \begin{bmatrix} s_{\eta,t}(\beta) \\ s_{\lambda,t}(\beta) \end{bmatrix}
\]

where

\[
\begin{align*}
    s_{\eta,t}(\beta) &= \left\{ \frac{v_t - x_t' \eta}{h_t} \right\} x_t - \left\{ \frac{[(v_t - x_t' \eta)^2 - h_t] \alpha (v_{t-1} - x_{t-1}' \eta)}{h_t^2} \right\} x_{t-1} \\
    s_{\lambda,t}(\beta) &= \left\{ \frac{(v_t - x_t' \eta)^2 - h_t}{2h_t^2} \right\} r_t(\eta)
\end{align*}
\]

where \( x_t = (1, v_{t-1})' \) and \( r_t(\eta) = |1, (v_{t-1} - x_{t-1}' \eta)^2|' \).
3. Consider the AR(1)-ARCH(1) model

\[
\begin{align*}
v_t &= \alpha_0 + \theta_0 v_{t-1} + h_t^{1/2} z_t \\
h_t &= \zeta_0 + \alpha_0 w_{t-1}^2 \\
w_t &= h_t^{1/2} z_t \\
z_t &\sim i.i.d.(\mu, \sigma^2_z)
\end{align*}
\]

Define \( \beta = (\eta', \lambda')' \) for \( \eta = (c, \theta)' \) and \( \lambda = (\zeta, \alpha)' \). Consider the quasi-MLE (QMLE) of \( \beta_0 \) based on the assumption that \( z_t \sim IN(0,1) \). It can be shown that (subject to certain mild regularity conditions) that the QMLE is consistent for \( \beta_0 \) provided \( E[s_t(\beta_0)|V_{t-1}] = 0 \) where \( s_t(\beta) \) is defined in Question 2. Show that this condition is satisfied if the true distribution of \( z_t \) satisfies \( \mu_z = 0 \) and \( \sigma^2_z = 1 \).