Exam 1

Answer all three questions. Please be sure to include all relevant working in the derivation of your answers. The marks for each question are given in bold in the left hand margin.

1. Consider the model

\[ y_t = \alpha_0 + \delta_0 t + u_t, \quad t = 1, 2, \ldots, T \]

Assume that \( u_t \sim i.i.d.(0, \sigma_0^2) \). Define \( D_T = \text{diag}(T^{1/2}, T^{3/2}) \), \( X \) to be the \( T \times 2 \) matrix with \( t^{th} \) row \((1, t)\) and \( u \) to be the \( T \times 1 \) vector with \( t^{th} \) element \( u_t \). Assume that \( D_T^{-1}X'u \overset{d}{\to} N(0, \sigma_0^2 Q) \) where \( \lim_{T \to \infty} D_T^{-1}X'XD_T^{-1} = Q \). Let \( \hat{\beta}_T = (\hat{\alpha}_T, \hat{\delta}_T)' \) be the OLS estimator of \( \beta_0 = (\alpha_0, \delta_0)' \).

\[ \begin{align*} 
15 \text{ (a) } & \text{Verify that } Q \text{ is finite and positive definite.} \\
20 \text{ (b) } & \text{Show that } D_T(\hat{\beta}_T - \beta_0) \overset{d}{\to} N(0, \sigma_0^2 Q^{-1}).
\end{align*} \]

2. Let \( \{v_t; \ t = 1, 2, \ldots\} \) be a weakly stationary scalar process with \( E[v_t] = 0 \) and \( E[v_t v_{t-j}] = \gamma_j \) for all \( t \) and \( j = 0, 1, 2, \ldots \). By definition,

\[ \text{Var}[T^{-1/2} \sum_{t=1}^{T} v_t] = \sum_{j=-T+1}^{T-1} \Gamma_T(j) \]

where

\[ \Gamma_T(j) = \begin{cases} T^{-1} \sum_{t=j+1}^{T} E[v_t v_{t-j}], & \text{for } j \geq 0 \\ T^{-1} \sum_{t=-j+1}^{T} E[v_t v_{t+j}], & \text{for } j < 0 \end{cases} \]

\[ \begin{align*}
15 \text{ (a) } & \text{Show that:} \\
20 \text{ (b) } & \text{Using part (a), show that if } \sum_{j=0}^{\infty} |\gamma_j| < \infty \text{ then } \lim_{T \to \infty} \text{Var}[T^{-1/2} \sum_{t=1}^{T} v_t] < \infty
\end{align*} \]

Continued over
3. Consider the linear regression model

\[ y = X\beta_0 + u \]  

(1)

in which (i) \( X \) is a \( T \times k \) matrix which is both fixed in repeated samples and has rank equal to \( k \); (ii) \( u \) is a \( T \times 1 \) random vector with \( E[u] = 0, \text{Var}[u] = \Omega \) whose \( t^{th} \) element is generated via:

\[ u_t = \theta_0 u_{t-1} + w_t \]  

(2)

where \( w_t \sim i.i.d.(0, \sigma_w^2) \) and \( |\theta_0| < 1 \). GLS estimation of this model is equivalent to OLS estimation of the transformed version of (1)

\[ \tilde{y} = \tilde{X}\beta_0 + \tilde{u} \]  

(3)

where \( \tilde{y} = Ny, \tilde{X} = NX, \tilde{u} = Nu \) and \( N \) is the matrix with \( i-j^{th} \) element defined as follows:

\[
N_{i,j} = \begin{cases} 
(1 - \theta_0^2)^{1/2}, & \text{for } i = j = 1 \\
-\theta_0, & \text{for } i = j + 1 \text{ and } j = 1, 2, \ldots, T - 1 \\
1, & \text{for } i = j \text{ and } j = 2, 3, \ldots, T \\
0, & \text{else.} 
\end{cases}
\]

Show that \( E[\tilde{u}] = 0 \) and \( \text{Var}[\tilde{u}] = \sigma_w^2 I_T \).

**Hint:** Within this model \( u_t = \sum_{i=0}^{\infty} \theta_0^i u_{t-i} \) and the \( j^{th} \) autocovariance of \( u_t \) is \( \theta_0^j \sigma_w^2 / (1 - \theta_0^2) \).