Solutions to Assessed Problem Set 2

1(a). The system can be written as:

\[
\begin{bmatrix}
1 - \theta_{1,1}L & -\theta_{1,2}L \\
-\theta_{2,1}L & 1 - \theta_{2,2}L
\end{bmatrix}
\begin{bmatrix}
v_{1,t} \\
v_{2,t}
\end{bmatrix}
= 
\begin{bmatrix}
w_{1,t} \\
w_{2,t}
\end{bmatrix}
\tag{1}
\]

From the definition of a matrix inverse, it follows that \(\text{adj}\{A(L)\}A(L) = \det\{A(L)\}I_2\) where \(\text{adj}\{A\}\) denotes the adjoint of the matrix \(A\), and \(\det\{A\}\) is the determinant of \(A\). Using this property of the adjoint, it follows from (2) that

\[
\text{adj}\{A(L)\}A(L)v_t = \text{adj}\{A(L)\}w_t
\]

\[
\det\{A(L)\}v_t = \text{adj}\{A(L)\}w_t
\tag{3}
\]

For this model, we have

\[
\det\{A(L)\} = (1 - \theta_{1,1}L)(1 - \theta_{2,2}L) - \theta_{1,2}\theta_{2,1}L^2
\]

\[
= 1 - (\theta_{1,1} + \theta_{2,2})L + (\theta_{1,1}\theta_{2,2} - \theta_{1,2}\theta_{2,1})L^2
\tag{4}
\]

and

\[
\text{adj}\{A(L)\} = 
\begin{bmatrix}
1 - \theta_{2,2}L & \theta_{1,2}L \\
\theta_{2,1}L & 1 - \theta_{1,1}L
\end{bmatrix}
\tag{5}
\]

Substituting (4)-(5) into (3) yields

\[
v_{1,t} = (\theta_{1,1} + \theta_{2,2})v_{1,t-1} - (\theta_{1,1}\theta_{2,2} - \theta_{1,2}\theta_{2,1})v_{1,t-2} + w_{1,t} - \theta_{2,2}w_{1,t-1} + \theta_{1,2}w_{2,t-1}
\tag{6}
\]

\[
v_{2,t} = (\theta_{1,1} + \theta_{2,2})v_{2,t-1} - (\theta_{1,1}\theta_{2,2} - \theta_{1,2}\theta_{2,1})v_{2,t-2} + w_{2,t} - \theta_{1,1}w_{2,t-1} + \theta_{2,1}w_{1,t-1}
\tag{7}
\]

1(b). Under the given assumptions, \(w_t \sim i.i.d.(0_{2 \times 1}, \Sigma)\) where the elements of \(\Sigma\) are:

\[
\Sigma = 
\begin{bmatrix}
\sigma_1^2 & \sigma_{1,2} \\
\sigma_{2,1} & \sigma_2^2
\end{bmatrix}
\]

where by construction \(\sigma_{1,2} = \sigma_{2,1}\). Therefore

\[
E[w_{i,t}w_{j,s}] = \sigma_i^2, \text{ for } i = j, t = s
\]

\[
= \sigma_{i,j}, \text{ for } i \neq j, t = s
\tag{8}
\]

\[
= 0, \text{ for } t \neq s
\tag{9}
\]
First note that we can define $e_{i,t}$ and $e_{2,t}$ via

$$e_{i,t} = w_{i,t} - \theta_{j,j}w_{i,t-1} + \theta_{i,j}w_{j,t-1}$$

for $i = 1, 2$ and $j \in \{1, 2\}$ but $j \neq i$.

From (11), it follows that

$$E[e_{i,t}] = E[w_{i,t}] - \theta_{j,j}E[w_{i,t-1}] + \theta_{i,j}E[w_{j,t-1}] = 0$$

because $E[w_t] = 0$ for all $t$. Recall that the $n^{th}$ autocorrelation of $e_{i,t}$ is defined to be $\rho_n^{(i)} = \gamma_n^{(i)}/\gamma_0^{(i)}$ where $\gamma_n^{(i)} = E[e_{i,t}e_{i,t-n}]$ (using $E|e_{i,t}| = 0$). Therefore, we first derive $\{\gamma_n^{(i)}\}$.

It follows from (11) that

$$\gamma_0^{(i)} = E[(w_{i,t} - \theta_{j,j}w_{i,t-1} + \theta_{i,j}w_{j,t-1})^2] = E[w_{i,t}^2] + \theta_{j,j}^2E[w_{i,t-1}^2] + \theta_{i,j}^2E[w_{j,t-1}^2] - 2\theta_{j,j}\theta_{i,j}E[w_{i,t}w_{j,t}]$$

Using (8)-(10) in (13), we obtain

$$\gamma_0^{(i)} = (1 + \theta_{j,j}^2)\sigma_i^2 + \theta_{i,j}^2\sigma_j^2 - 2\theta_{j,j}\theta_{i,j}\sigma_{i,j}$$

It follows from (11) that

$$\gamma_1^{(i)} = E[(w_{i,t} - \theta_{j,j}w_{i,t-1} + \theta_{i,j}w_{j,t-1})(w_{i,t-1} - \theta_{j,j}w_{i,t-2} + \theta_{i,j}w_{j,t-2})] = E[w_{i,t}w_{i,t-1}] - \theta_{j,j}E[w_{i,t}w_{j,t-2}] + \theta_{i,j}E[w_{i,t}w_{j,t-1}] - \theta_{j,j}\theta_{i,j}E[w_{i,t}w_{j,t-1} - \theta_{j,j}\theta_{i,j}E[w_{i,t}w_{j,t-1}]] + \theta_{i,j}^2E[w_{j,t-1}w_{j,t-2}]$$

Using (8)-(10) in (15), we obtain

$$\gamma_1^{(i)} = \theta_{i,j}\sigma_{i,j} - \theta_{j,j}\sigma_i^2$$

It follows from (11)

$$\gamma_n^{(i)} = E[(w_{i,t} - \theta_{j,j}w_{i,t-1} + \theta_{i,j}w_{j,t-1})(w_{i,t-n} - \theta_{j,j}w_{i,t-n-1} + \theta_{i,j}w_{j,t-n-1})] = E[w_{i,t}w_{i,t-n}] - \theta_{j,j}E[w_{i,t}w_{j,t-n}] + \theta_{i,j}E[w_{i,t}w_{j,t-n-1}] - \theta_{j,j}\theta_{i,j}E[w_{i,t}w_{j,t-n-1}]$$

Using (8)-(10) in (17), it follows that for $n > 1$

$$\gamma_n^{(i)} = 0$$
It follows from (14), (16) and (18) that the autocorrelation function of $e_{i,t}$ is:

$$
\rho^{(i)}_n = 1, \quad \text{for } n = 0
$$

$$
= \frac{\theta_{i,j}\sigma_{i,j} - \theta_{j,j}\sigma^2_i}{(1 + \theta^2_{j,j})\sigma^2_i + \theta^2_{i,j}\sigma^2_j - 2\theta_{i,j}\theta_{j,j}\sigma_{i,j}}, \quad \text{for } n = 1
$$

$$
= 0, \quad \text{for } n > 1
$$

Notice that this autocorrelation function exhibits a cut-off after lag 1 in general (i.e. if $\theta_{i,j}\sigma_{1,2} \neq \theta_{j,j}\sigma^2_i$) which is also the pattern exhibited by a $MA(1)$ process.

1(c) $p = 2, q = 1$.

1(d) The condition for stationarity of an $ARMA(p, q)$ process is that the roots of the AR polynomial lie outside the unit circle. The condition for stationarity of a $VAR(p)$ process is that the roots of $det\{A(m)\} = 0$ lie outside the unit circle. In this question, it can be seen form (3) that the AR polynomial in each univariate representation is $det\{A(L)\}$ and so the conditions for stationarity are equivalent.

In fact, the property demonstrated in this question generalizes. If $v_t$ is $k \times 1$ and has a stationary $VAR(p)$ representation then each element of $v_t$ has a stationary $ARMA(m, n)$ representation where $m = kp$ and $n = (k - 1)p$