Assessed Problem Set 3

Answer all questions. Your answers must be handed in to me in class on Wednesday, April 13.

Consider the model

\[ y_t = \theta_0 y_{t-1} + m_t' \gamma_0 + u_t \]
\[ u_t = \rho_0 u_{t-1} + w_t \] (1)

where \( \beta = (\theta, \gamma)' \), \( x_t' = (y_{t-1}, m_t') \), \( m_t \) is \( k \times 1 \), \(|\theta_0| < 1\), \(|\rho_0| < 1\), and \( w_t \sim iid(0, \sigma^2) \) with \( \sigma^2 > 0 \).

You may assume:

(a) \( \{m_t\} \) is independent of \( \{u_t\} \);
(b) \( v_t = (m_t, u_t)' \) is covariance stationary and ergodic for the second moments;
(c) \( y_t = \sum_{i=0}^{\infty} \theta_i a_{t-i} \) where \( a_t = m_t' \gamma_0 + u_t \);
(d) \( E[x_t x_t'] = Q \), a nonsingular matrix of constants.

Note that (b) and (c) imply that \( (x_t, u_t)' \) is covariance stationary and ergodic for the second moments.

1. Let \( \hat{\beta}_{OLS} \) be the OLS estimator of \( \beta_0 \) based on (1). Show that \( \hat{\beta}_{OLS} \) is an inconsistent estimator if \( \rho_0 \neq 0 \) but that \( \hat{\beta}_{OLS} \) is a consistent estimator for \( \beta_0 \) if \( \rho_0 = 0 \).

2. Let \( z_t = (m_{i,t-1}, m_t)' \) where \( m_{i,t-1} \) is the \( i^{th} \) element of \( m_{t-1} \) and consider the Instrumental Variables (IV) estimator of \( \beta_0 \), \( \hat{\beta}_{IV} = (Z'X)^{-1}Z'y \) where \( Z \) is the \( T \times (k + 1) \) matrix with \( t^{th} \) row \( z_t \). Assuming that \( E[z_t x_t]' = Q_{zx} \), a nonsingular matrix, show that \( \hat{\beta}_{IV} \) is consistent for \( \beta_0 \).

3. Assuming that \( T^{-1/2} Z' u \) satisfies the CLT for covariance stationary processes, show that \( T^{1/2}(\hat{\beta}_{IV} - \beta_0) \xrightarrow{d} N(0, Q_{zx}^{-1} S(Q_{zx}^{-1})') \) where \( S \) is the long run variance of \( z_t u_t \).

4. Suppose it is desired to test the null hypothesis \( H_0 : R\beta_0 = r \) where \( R \) is a \( q \times (k + 1) \) matrix of constants with rank equal to \( q \) and \( r \) is a \( q \times 1 \) vector of constants. Propose a test statistic based on \( \hat{\beta}_{IV} \) and state its distribution under the null hypothesis.