MA 401 - Fall 2012 - Test # 1 – Solution

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1. True or false? Justify your answer! The correct answer without a proper justification will give you NO credit at all.

(a) (10 pts.) Let \( u_1(x,t) \) and \( u_2(x,t) \) be solutions of the equation \( u_t - u_{xx} + xu = 1 \). Then the function \( v(x,t) = C_1u_1(x,t) + C_2u_2(x,t) \) is also a solution of this equation for any constants \( C_1 \) and \( C_2 \).

**Solution:** False, since this linear equation is nonhomogeneous (the superposition principle holds only for linear homogeneous equations).

(b) (10 pts.) Functions \( \{\sin x, \sin 3x, \sin 5x, \ldots\} \) form an orthogonal family on \( 0 < x < \frac{\pi}{2} \).

**Solution:** True, since the inner products
\[
\int_0^{\pi/2} \sin((2n-1)x) \sin((2m-1)x) \, dx = 0, \quad \forall m, n = 1, 2, \ldots, \quad m \neq n.
\]

2. (15 pts.) What is the type of each of the following equations? State whether the equation is homogeneous or inhomogeneous?

(a) \( u_{xx} - 6u_{xy} + 10u_{yy} + u_x - 3u_y = 0 \).

(b) \( 9u_{xx} + 6u_{xy} + u_{yy} + u_x - 1 = 0 \).

(c) \( u_{xx} - 4u_{xy} + 2u_y + u_{yy} + 4u = 0 \).

**Solution:**

(a) \( u_{xx} - 6u_{xy} + 10u_{yy} + u_x - 3u_y = 0 \) \( \implies \) \( A = 1, \ B = -6, \ C = 10, \ B^2 - 4AC = -4 < 0 \), elliptic, homogeneous.

(b) \( 9u_{xx} + 6u_{xy} + u_{yy} + u_x - 1 = 0 \) \( \implies \) \( A = 9, \ B = 6, \ C = 1, \ B^2 - 4AC = 0 \), parabolic, inhomogeneous.
(c) \( u_{xx} - 4u_{xy} + 2u_y + u_{yy} + 4u = 0 \implies A = 1, \ B = -4, \ C = 1, \ B^2 - 4AC = 12 > 0, \) hyperbolic, homogeneous.

3. Consider Robin-Dirichlet boundary-value problem on \( 0 < x < 1: \)

\[
X'' + \lambda X = 0, \quad X'(0) + X(0) = 0, \quad X(1) = 0.
\]

(a) (5 pts.) Are the boundary conditions symmetric? Justify your answer.

**Solution:** Yes, since for any pair of functions \( X_1 \) and \( X_2 \) satisfying the problem, we have \( X'(0) = -X(0), \ X(1) = 0 \) and thus

\[
X'_1(1)X_2(1) - X_1(1)X'_2(1) - (X'_1(0)X_2(0) - X_1(0)X'_2(0)) = 0 - 0 - (-X_1(0)X_2(0) + X_1(0)X_2(0)) = 0.
\]

(b) (5 pts.) Are all the eigenvalues of the problem real? Justify your answer.

**Solution:** Yes, since the considered eigenvalue problem has symmetric boundary conditions (see Theorem 2.2 on page 53).

(c) (5 pts.) Is it possible to conclude without direct computation that all the eigenvalues are nonnegative? Justify your answer.

**Solution:** According to Theorem 2.3 on page 54, the answer is "No", since

\[
X'(1)X(1) - X(0)X'(0) = 0 + X^2(0) \geq 0.
\]

(d) (10 pts.) Show by direct computation that \( \lambda = 0 \) is an eigenvalue. Find the corresponding eigenfunction.

**Solution:** When \( \lambda = 0 \), the equation becomes \( X'' = 0 \), with general solution \( X(x) = Ax + B \). Since \( X(1) = 0 \), we have \( A = -B \), so that \( X(x) = Ax - A \). Then \( X'(0) + X(0) = A - A = 0 \) is always satisfied. Therefore, \( \lambda = 0 \) is an eigenvalue with eigenfunction \( X_0(x) = x - 1 \) (determined up to a constant multiple).

(e) (10 pts.) Show by direct computation that there no negative eigenvalues.

**Solution:** If \( \lambda < 0 \), the general solution of the equation is \( X(x) = A \cosh(\sqrt{\lambda}x) + B \sinh(\sqrt{\lambda}x) \). The boundary condition \( X'(0) + X(0) = 0 \) implies that \( A = -B\sqrt{\lambda} \); then \( X(1) = 0 \) implies that \( -B\sqrt{\lambda}\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x) = B(\sinh(\sqrt{\lambda}x) - \sqrt{\lambda}\cosh(\sqrt{\lambda}x))) = 0 \). Rejecting \( B = 0 \) since it would give the trivial solution, we have \( \sqrt{\lambda} = \tanh(\sqrt{\lambda}) \). The last equation has only the solution \( \lambda = 0 \) (which is impossible since we are assuming that \( \lambda < 0 \)). Thus, no negative eigenvalues.
(f) (10 pts.) Find an equation for all positive eigenvalues. **Do not solve the equation.**

**Solution:** If \( \lambda > 0 \), the general solution of the equation is \( X(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x) \). The boundary condition \( X'(0) + X(0) = 0 \) implies that \( A = -B\sqrt{\lambda} \); then \( X(1) = 0 \) implies that \( -B\sqrt{\lambda} \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x) = B(\sin(\sqrt{\lambda}x) - \sqrt{\lambda} \cos(\sqrt{\lambda}x)) = 0 \). Rejecting \( B = 0 \) since it would give the trivial solution, we have \( \sqrt{\lambda} = \tan(\sqrt{\lambda}) \). The last equation has infinite number of solutions for \( \lambda > 0 \).

4. Consider the initial-boundary value problem

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, \ t > 0, \\
u(x,0) = f(x), \\
u_t(x,0) = g(x), \\
u(0,t) = u(\pi,t) = 0.
\end{cases}
\]

(a) (10 pts.) Write the general solution of this initial-boundary value problem.

**Solution:** This is a \( D - D \) initial boundary value problem with \( c = 1, \ l = \pi \), so we have

\[
u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)) \sin(nx).
\]

(b) (10 pts.) Suppose \( f(x) = 0 \) and \( g(x) = \sin(401x) - 4 \sin(2012x) \). Find the coefficients in the general solution you found in part (a).

**Solution:** With \( f(x) = 0 \) and \( g(x) = \sin(401x) - 4 \sin(2012x) \), we have

\[
u(x,0) = 0 = \sum_{n=1}^{\infty} a_n \sin(nx),
\]

\[
u_t(x,0) = \sin(401x) - 4 \sin(2012x) = \sum_{n=1}^{\infty} nb_n \sin(nx).
\]

The coefficients \( a_n = 0, \ \forall n = 1, 2, \ldots \) since

\[
 a_n = \frac{2}{\pi} \int_0^{\pi} 0 \sin(nx) \, dx = 0,
\]

while \( b_n \) can be computed using

\[
b_n = \frac{2}{\pi} \int_0^{\pi} (\sin(401x) - 4 \sin(2012x)) \sin(nx) \, dx
\]

\[
= \frac{2}{\pi} \int_0^{\pi} \sin(401x) \sin(nx) \, dx - \frac{8}{\pi} \int_0^{\pi} \sin(2012x) \sin(nx) \, dx.
\]
Taking into account the orthogonality property of the sine functions, we have

\[ \int_0^\pi \sin(401x) \sin(nx) \, dx = \begin{cases} 
0, & n \neq 401, \\
\pi/2, & n = 401,
\end{cases} \]

and

\[ \int_0^\pi \sin(2012x) \sin(nx) \, dx = \begin{cases} 
0, & n \neq 2012, \\
\pi/2, & n = 2012.
\end{cases} \]

Then,

\[ nb_n = \begin{cases} 
1, & n = 401, \\
-4, & n = 2012, \\
0, & \text{otherwise},
\end{cases} \]

Finally, since we have \( a_n = 0 \) and

\[ b_n = \begin{cases} 
1/n, & n = 401, \\
-4/n, & n = 2012, \\
0, & \text{otherwise},
\end{cases} \]

the general solution of the problem is given by

\[ u(x, t) = \frac{1}{401} \sin(401t) \sin(401x) - \frac{4}{2012} \sin(2012t) \sin(2012x). \]