Exercise 2.2.4

(a)
- The first line is the wave equation with wave speed $c$.
- The boundary condition $u_x(O, t) = 0$ means that the slope of the string at its left end is zero at all times.
- The boundary condition $u(l, t) = 0$ means that the right end of the string is held at zero height at all times.
- The initial condition $u(x, O) = f(x)$ means that the shape of the string at time 0 is given by $f(x)$.
- The initial condition $u_t(x, O) = g(x)$ means that at time 0, the velocity of the string is $g(x)$.

(b) Assuming a solution of the form $u(X, T)$, we have $u_{tt} = T''(t)X(x)$ and $u_{xx} = T(t)X''(x)$, so that $T''(t)X(x) = c^2T(t)X''(x)$. Collecting terms gives
\[
\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda,
\]
where $\lambda$ is defined to be the negative of the common value of the two formulas. This gives us two ODEs:
\[
X''(x) + \lambda X(x) = 0, \quad T''(t) + c^2\lambda T(t) = 0.
\]
Considering the ODE for $X$, the boundary conditions are $u_x(O, t) = X'(O)T(t) = 0$ and $u(l, t) = X(l)T(t) = 0$. Since we are looking for nontrivial solutions, $T(t) \neq 0$, so that we have $X'(0) = 0$ and $X(l) = 0$. The eigenvalue problem is thus
\[
X''(x) + \lambda X(x) = 0, \quad X'(0) = 0, \quad X(l) = 0.
\]

(c) To solve the $X$ problem, there are three cases:

- Case 1: $\lambda = 0$. The equation becomes $X''(x) = 0$, whose general solution is $X = Ax + B$ where $A$ and $B$ are arbitrary constants. $0 = X'(O) = A$ implies that $A = 0$, and then $0 = X(l) = B$ means that $B = 0$ as well. Thus $0$ is not an eigenvalue.

- Case 2: $\lambda < 0$. The equation is then $X'' + \lambda X = 0$, whose general solution is $X = A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x)$ with $X' = A\sqrt{\lambda}\sinh(\sqrt{\lambda}x) + B\sqrt{\lambda}\cosh(\sqrt{\lambda}x)$. Then $0 = X'(0) = A\sqrt{\lambda}\sinh 0 + B\sqrt{\lambda}\cosh 0 = B\sqrt{\lambda}$ implies that $B = 0$. The other boundary condition gives $0 = X(l) = A\cosh(\sqrt{\lambda}l)$. But $\cosh(x) > 0$ for all $x$, so we must have $A = 0$ as well so that there are no negative eigenvalues.

- Case 3: $\lambda > 0$. The equation is $X'' + \lambda X = 0$, whose general solution is $X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$ with $X' = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$. The boundary conditions give us $X'(0) = -A\sqrt{\lambda}\sin 0 + B\sqrt{\lambda}\cos 0 = B\sqrt{\lambda} = 0$ so that $B = 0$, and also $X(l) = A\cos(\sqrt{\lambda}l) = 0$. We thus get a nontrivial solution whenever $\cos(\sqrt{\lambda}l) = 0$, i.e., when $\sqrt{\lambda}l = \frac{(2n - 1)\pi}{2}$ for
\( n = 1, 2, 3, \ldots \) (note that \( n \) must be positive since both \( \sqrt{\lambda} \) and \( l \) are positive). Thus the eigenvalues are

\[
\lambda_n = \left( \frac{(2n - 1)\pi}{2l} \right)^2, \quad n = 1, 2, 3, \ldots,
\]

with corresponding eigenfunctions

\[
X_n(x) = \cos \left( \frac{(2n - 1)\pi}{2l} x \right)
\]

determined up to a constant multiple.

(d) The ODE for \( T \) is now

\[
T_n''(t) + c^2 \lambda_n T_n(t) = 0,
\]

and since both \( c^2 \) and \( \lambda_n \) are positive, the general solution for this equation is

\[
T_n(t) = A \cos(c\sqrt{\lambda_n}t) + B \sin(c\sqrt{\lambda_n}t),
\]

which we write as

\[
T_n(t) = a_n \cos(c\sqrt{\lambda_n}t) + b_n \sin(c\sqrt{\lambda_n}t).
\]

(e) We now have a family of solutions \( u_n(x, t) = T_n(t)X_n(x) \) for the original boundary value problem; by the superposition principle, the general solution is given by an arbitrary linear combination of the \( u_n(x, t) \), i.e.,

\[
\begin{align*}
    u(x, t) &= \sum_{n=1}^{\infty} T_n(t)X_n(x) \\
            &= \sum_{n=1}^{\infty} \left[ a_n \cos(c\sqrt{\lambda_n}t) + b_n \sin(c\sqrt{\lambda_n}t) \right] \cos(\sqrt{\lambda_n}x) \\
            &= \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{c(2n - 1)\pi}{2l} t \right) + b_n \sin \left( \frac{c(2n - 1)\pi}{2l} t \right) \right] \cos \left( \frac{(2n - 1)\pi}{2l} x \right).
\end{align*}
\]

Exercise 2.3.4

Let

\[
u(x, t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos(\sqrt{\lambda_n}x) + b_n \sin(\sqrt{\lambda_n}x) \right] e^{-\lambda_n kt},\]

with initial condition \( u(x, 0) = f(x) \). Then

\[
f(x) = u(x, 0) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos(\sqrt{\lambda_n}x) + b_n \sin(\sqrt{\lambda_n}x) \right]. \tag{1}
\]

Note first that simply integrating both sides gives (remembering that \( \lambda_n = \left( \frac{n\pi}{l} \right)^2 \))

\[
\int_{-l}^{l} f(x) \, dx = \int_{-l}^{l} \left[ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi}{l} x \right) + b_n \sin \left( \frac{n\pi}{l} x \right) \right] \right] \, dx = a_0 l.
\]
so that
\[ a_0 = \frac{1}{l} \int_{-l}^{l} f(x) \, dx. \]

To compute \( a_m \) for \( m \neq 0 \), multiply (1) through by \( \cos \left( \frac{m\pi}{l}x \right) \) and integrate (using the results of Exercise 3 in this Section) to get
\[
\int_{-l}^{l} f(x) \cos \left( \frac{m\pi}{l}x \right) \, dx = \frac{1}{l} a_0 \cos \left( \frac{m\pi}{l}x \right) \, dx + \sum_{n=1}^{\infty} \int_{-l}^{l} a_n \cos \left( \frac{n\pi}{l}x \right) \cos \left( \frac{m\pi}{l}x \right) \, dx
\]
\[+ \int_{-l}^{l} b_n \sin \left( \frac{n\pi}{l}x \right) \cos \left( \frac{m\pi}{l}x \right) \, dx = a_m l, \]
so that
\[ a_m = \frac{1}{l} \int_{-l}^{l} f(x) \cos \left( \frac{m\pi}{l}x \right) \, dx. \]

Similarly, to compute \( b_m \) for \( m \neq 0 \), multiply (1) through by \( \sin \left( \frac{m\pi}{l}x \right) \) and integrate to obtain
\[
\int_{-l}^{l} f(x) \sin \left( \frac{m\pi}{l}x \right) \, dx = \frac{1}{l} a_0 \sin \left( \frac{m\pi}{l}x \right) \, dx + \sum_{n=1}^{\infty} \int_{-l}^{l} a_n \cos \left( \frac{n\pi}{l}x \right) \sin \left( \frac{m\pi}{l}x \right) \, dx
\]
\[+ \int_{-l}^{l} b_n \sin \left( \frac{n\pi}{l}x \right) \sin \left( \frac{m\pi}{l}x \right) \, dx = b_m l, \]
so that
\[ b_m = \frac{1}{l} \int_{-l}^{l} f(x) \sin \left( \frac{m\pi}{l}x \right) \, dx. \]

**Exercise 2.4.2**

Suppose \( X_1 \) and \( X_2 \) both satisfy the periodic boundary conditions \( X(a) = X(b), \ X'(a) = X'(b) \). Then
\[-X'_1(b)X_2(b) + X_1(b)X'_2(b) + X'_1(a)X_2(a) - X_1(a)X'_2(a) = -X'_1(b)X_2(b) + X_1(b)X'_2(b) + X'_1(b)X_2(b) - X_1(b)X'_2(b) = 0. \]

**Exercise 2.5.2**

(a) The first line is the standard heat equation. The second and third lines are Robin boundary conditions both with ambient temperature zero and \( K = 1 \), so they both specify physically realistic convection with a surrounding medium fixed at 0. The fourth line gives the initial temperature distribution of the wire.
(b) Following the usual method, we get the two equations
\[ X'' + \lambda X = 0, \quad T' + \lambda T = 0. \]
The boundary conditions give initial conditions for the spatial problem as follows:
\[ u_x(0, t) - u(0, t) = 0 \implies X'(0) - X(0) = 0, \]
\[ u_x(1, t) + u(1, t) = 0 \implies X'(1) + X(1) = 0. \]

(c) The boundary conditions are easily verified to be symmetric (see (2.28) and (2.29)). Theorem 2.3 tells us that if every solution satisfies
\[ X'(x)X(x) \bigg|_0^1 \leq 0, \]
then there are no negative eigenvalues, and indeed
\[ X'(x)X(x) \bigg|_0^1 = X'(1)X(1) - X'(0)X(0) = -X(1)^2 - X(0)^2 \leq 0. \]
To show that zero is not an eigenvalue, suppose \( \lambda = 0 \); then we have \( X''(x) = 0 \) so that \( X = Ax + B \); then \( X'(0) - X(0) = A - B = 0 \) so that \( A = B \), but also \( X'(1) + X(1) = 2A + B = 0 \). Thus \( 3A = 0 \) so \( A = B = 0 \). Hence all eigenvalues are positive.

(d) Let \( \lambda > 0 \) be an eigenvalue. The general solution to \( X'' + \lambda X = 0 \) is \( X = A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x) \). The first boundary condition gives \( X'(0) - X(0) = B\sqrt{\lambda} - A = 0 \) so that \( A = B\sqrt{\lambda} \). Making this substitution to write \( X = B\sqrt{\lambda} \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x) \) and applying the second boundary condition gives
\[ X'(1) + X(1) = (-B\lambda \sin(\sqrt{\lambda}) + B\sqrt{\lambda} \cos(\sqrt{\lambda})) = B((1 - \lambda) \sin(\sqrt{\lambda}) + 2\sqrt{\lambda} \cos(\sqrt{\lambda})) = 0. \]
Divide this equation through by \( B \cos(\sqrt{\lambda}) \) to get \( (1 - \lambda) \tan(\sqrt{\lambda}) + 2\sqrt{\lambda} = 0 \). Treating this as a quadratic equation in \( \sqrt{\lambda} \) and solving gives
\[ \sqrt{\lambda} = \frac{\cos(\sqrt{\lambda}) \pm 1}{\sin(\sqrt{\lambda})}. \]
So positive solutions to these two equations are the eigenvalues. For each such eigenvalue, the corresponding eigenfunction is (up to a constant multiple) \( X_\lambda(x) = \sqrt{\lambda} \cos(\sqrt{\lambda} x) + \sin(\sqrt{\lambda} x) \).

(e) For each such \( \lambda \), the \( T \) problem is
\[ T' + \lambda T = 0, \]
so that \( T_\lambda(t) = e^{-\lambda t} \) up to a constant multiple.

(f) Numbering the (countable number of) positive eigenvalues \( \lambda_1, \lambda_2, \ldots \), we have as a general solution
\[ u(x, t) = \sum_{n=1}^{\infty} a_n \left[ \sqrt{\lambda_n} \cos(\sqrt{\lambda_n} x) + \sin(\sqrt{\lambda_n} x) \right] e^{-\lambda_n t}. \]

(g) We have
\[ u(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n \left[ \sqrt{\lambda_n} \cos(\sqrt{\lambda_n} x) + \sin(\sqrt{\lambda_n} x) \right]. \]
The condition of part (b) of Theorem 2.1 is satisfied since the integration interval is finite and $f$ is assumed continuous. Thus we can compute

$$a_n = \frac{\int_0^1 f(x) \left( \sqrt{\lambda_n} \cos(\sqrt{\lambda_n}x) + \sin(\sqrt{\lambda_n}x) \right) \, dx}{\int_0^1 \left( \sqrt{\lambda_n} \cos(\sqrt{\lambda_n}x) + \sin(\sqrt{\lambda_n}x) \right)^2 \, dx}.$$ 

The denominator can be computed (in terms of $\lambda_n$), but is quite complicated.