Exercise 2.1.4

(a) We have
\[ \langle f, g \rangle = \int_0^l f(x)g(x) \, dx = \int_0^l (x - x^3) \, dx = \frac{2l^2 - l^4}{4}. \]
Thus, \( f \perp g \) when \( 2l^2 = l^4 \). For \( l > 0 \), this occurs for \( l = \sqrt{2} \).

(b) \[ \|f\| = \sqrt{\int_0^l f^2(x) \, dx} = \sqrt{\int_0^l x^2 \, dx} = \left( \frac{8}{9} \right)^{1/4}. \]

Exercise 2.1.8

(a) Let \( \mathcal{L} u = -x^2 u'' - xu' \); then the eigenvalue problem is
\[ \mathcal{L} u = \lambda u, \quad u(1) = 0, \quad u(5) = 0. \]

(b) If \( \lambda = 0 \), the problem reduces to
\[ x^2 y'' + xy' = 0, \quad y(1) = 0, \quad y(5) = 0. \]
This is a Cauchy-Euler equation with \( a = b = 1 \) and \( c = 0 \); the associated characteristic equation is
\[ r^2 + (b - 1)r + c = r^2 = 0, \]
which has the repeated real root 0. Thus the general solution to this equation is
\[ y = A + B \ln x. \]
The boundary conditions give the simultaneous equations \( A = 0, A + B \ln 5 = 0 \), so that \( A = B = 0 \). Thus the only particular solution is the trivial solution, so that zero is not an eigenvalue.

(c) If \( \lambda < 0 \), then we have
\[ x^2 y'' + xy' - \lambda y = 0, y(1) = 0, y(5) = 0. \]
The characteristic equation is \( r^2 - \lambda = 0 \), which has the two real roots \( r_{1,2} = \pm \sqrt{\lambda} \). The general solution is thus
\[ y = A x^{\sqrt{\lambda}} + B x^{-\sqrt{\lambda}}. \]
The boundary conditions give:
\[ O = y(1) = A + B, \quad 0 = y(5) = A 5^{\sqrt{\lambda}} + B 5^{-\sqrt{\lambda}}, \]
which has only the solution \( A = B = 0 \). So there are no negative eigenvalues either.
(d) If \( \lambda > 0 \), then we have, \( r_{1,2} = \pm i\sqrt{\lambda} \). The general solution is thus
\[
y = A \cos \left( \sqrt{\lambda} \ln x \right) + B \sin \left( \sqrt{\lambda} \ln x \right).
\]
The boundary conditions give:
\[
O = y(1) = A, \quad 0 = y(5) = A \cos \left( \sqrt{\lambda} \ln 5 \right) + B \sin \left( \sqrt{\lambda} \ln 5 \right),
\]
so that \( A = 0 \) and \( B \sin \left( \sqrt{\lambda} \ln 5 \right) = 0 \). Rejecting \( B = 0 \), we must have \( \sin \left( \sqrt{\lambda} \ln 5 \right) = 0 \), so that
\[
\sqrt{\lambda} \ln 5 = \pm n\pi, \quad n = 1, 2, 3 \ldots,
\]
and then
\[
\lambda_n = \frac{n^2 \pi^2}{(\ln 5)^2}
\]
with corresponding eigenfunctions (up to a constant multiple)
\[
y_n(x) = \sin \left( \frac{n\pi}{\ln 5} \ln x \right).
\]

**Exercise 2.2.2**

(a) For Examples 2.2.1 and 2.2.3, the time ODE was first-order, so will have a one-parameter family of solutions. The time ODE in Example 2.2.2 was second-order and has a two-parameter family of solutions. These parameters correspond precisely to the indexed families of constants.

(b) In each of these cases, there is a family of eigenvalues that arise as solutions to the spatial ODE. Each of these eigenvalues gives rise to an eigenfunction for the entire system. By the superposition principle, then, the general solution is an arbitrary linear combination of these eigenfunctions. The coefficients in that linear combination are the families of constants.

**Exercise 2.3.8**

(a) The general solution we found in Exercise 2.2.3 was
\[
u(x, t) = \sum_{n=1}^{\infty} c_n \sin \left( \sqrt{\lambda_n} x \right) e^{-\sqrt{\lambda_n} kt} = \sum_{n=1}^{\infty} c_n \sin \left( \frac{(2n - 1)\pi x}{2l} \right) e^{-\frac{(2n - 1)\pi kt}{2l}}.
\]

Then
\[
f(x) = u(x, 0) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{(2n - 1)\pi x}{2l} \right),
\]
so that \( \{c_n\} \) are the Fourier sine series coefficients for \( f(x) \), and thus
\[
c_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \left( \frac{(2n - 1)\pi x}{2l} \right) \, dx.
\]
(b) With $k = 2, l = 1$, and $f(x) = 180x(1 - x)^4$, we have

$$c_n = 2 \int_0^1 180x(1 - x)^4 \sin \left( \frac{(2n - 1) \pi x}{2} \right) \, dx.$$ 

From the equation for $u(x,t)$, we would expect the solution to decay to zero as the exponential decay terms take over.