Exercise 1.1.6

With \( u(x, y) = e^{-y}(\sin x + \cos x) \), we have

\[
\begin{align*}
    u_x &= e^{-y}(\cos x - \sin x), \quad u_{xx} = e^{-y}(-\sin x - \cos x), \\
    u_x &= -e^{-y}(\sin x + \cos x), \quad u_{xx} = e^{-y}(\sin x + \cos x),
\end{align*}
\]

so that \( u_{xx} + u_{yy} = 0 \). Further, \( u(x, 0) = \sin x + \cos x \). Finally, \( u(0, y) = u_x(0, y) = e^{-y} \).

Exercise 1.3.2

The PDE itself is the heat equation,

\[
    u_t - ku_{xx} = 0.
\]

At the right boundary, we have a Dirichlet boundary condition which says that the temperature at time \( t \) is \( f(t) \); thus the boundary condition is \( u(l, t) = f(t) \). At the left boundary, we have a Robin boundary condition; the relevant formula is \( u_x = K(u - T/K) \), where \( T/K \) is the ambient temperature. Here, \( T/K = M(t) \), so the boundary condition is \( u_x(0, t) = K(u(0, t) - M(t)) \).

Exercise 1.4.4

(a) Since \( \varphi \) solves \( \mathcal{L} u = 0 \), we have \( \mathcal{L} \varphi = 0 \). Since \( \psi \) solves \( \mathcal{L} u = f \), we have \( \mathcal{L} \psi = f \). But \( \mathcal{L} \) is linear by assumption, so \( \mathcal{L}(\varphi + \psi) = \mathcal{L} \varphi + \mathcal{L} \psi = f \), and thus \( \varphi + \psi \) also solves the nonhomogeneous problem \( \mathcal{L} u = f \).

(b) No.

Exercise 1.4.6

(a) To find the general solution, use the integrating factor

\[
m(t) = e^{\int 2t \, dy} = e^{t^2}.
\]

Multiplying the equation through by \( m(t) \) yields

\[
    e^{t^2} y' + 2te^{t^2} y = te^{t^2},
\]

\[
    (e^{t^2} y)' = te^{t^2}.
\]

Integrating both sides yields

\[
e^{t^2} y = \frac{1}{2} e^{t^2} + C \quad \Rightarrow \quad y = \frac{1}{2} + Ce^{-t^2}.
\]

(b) Substituting the values from the initial condition \( y(0) = \sqrt{2} \), gives

\[
    \sqrt{2} = \frac{1}{2} + C \quad \Rightarrow \quad C = \sqrt{2} - \frac{1}{2}.
\]
Thus, the particular solution is
\[ y = \frac{1}{2} + \left( \sqrt{2} - \frac{1}{2} \right) e^{-t^2}. \]

Exercise 1.4.8

(a) Here, the roots of the characteristic equation are
\[ r_{1,2} = \frac{-4 \pm 6}{2} = -5, 1, \]
so that the general solution is
\[ y = C_1 e^{-5t} + C_2 e^t. \]

(b) Note that \( y' = -5C_1 e^{-5t} + C_2 e^t \). Then the conditions \( y(0) = 1, y'(0) = 0 \) give the two simultaneous equations
\[ C_1 + C_2 = 1, \quad -5C_1 + C_2 = 0, \]
with solutions \( C_1 = 1/6, \ C_2 = 5/6, \) so that the particular solution is
\[ y = \frac{1}{6} e^{-5t} + \frac{5}{6} e^t. \]

Exercise 1.4.13(a)

This equation is of the form
\[ ax^2 y''(x) + by'(x) + cy(x) = 0, \]
with \( a = 1, b = 2, c = -6 \). Solving the characteristic equation
\[ ap^2 + (b - 1)p + c = 0, \quad \text{or} \quad p^2 + p - 6 = 0, \]
we obtain \( p = 2 \) and \( p = -3 \). Therefore, the general solution is
\[ y = C_1 x^2 + C_2 x^{-3}. \]