The Definite Integral

**Definition of a Definite Integral** If $f$ is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into $n$ subintervals of equal width $\Delta x = (b - a)/n$. We let $a = x_0, x_1, x_2, \ldots, x_n = b$ be the endpoints of these subintervals and we choose sample points $x_1^*, x_2^*, \ldots, x_n^*$ in these subintervals, so $x_i^*$ lies in the $i$th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of $f$ from $a$ to $b$** is

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

**Remarks.**

1. Because we have assumed that $f$ is continuous, it can be proved that the limit in Definition always exists and gives the same value no matter how we choose the sample point $x_i^*$.

2. The symbol $\int$ was introduced by Leibnitz and is called an **integral sign**. In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand** and $a$ and $b$ are called the **limits of integration**: $a$ is the **lower limit** and $b$ is the **upper limit**. The procedure of calculating an integral is called **integration**.

3. The definite integral $\int_a^b f(x)dx$ is a number; it does not depend on $x$. In fact, we could use any letter in place of $x$ without changing the value of the integral:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

**Properties of the Definite Integral**

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

1. $\int_a^b cdx = c(b - a)$, where $c$ is any constant.

2. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where $c$ is any constant.

4. $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
5. If \( f(x) \geq 0 \) for \( a \leq x \leq b \), then \( \int_{a}^{b} f(x) \, dx \geq 0 \).

6. If \( f(x) \geq g(x) \) for \( a \leq x \leq b \), then \( \int_{a}^{b} f(x) \, dx \geq \int_{a}^{b} g(x) \, dx \).

7. If \( m \leq f(x) \leq M \) for \( a \leq x \leq b \), then
   \[
   m(b-a) \leq \int_{a}^{b} f(x) \, dx \leq M(b-a).
   \]

**The Fundamental Theorem of Calculus** Suppose \( f \) is continuous on \([a, b]\).

1. \( \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x) \).

2. \( \int_{a}^{b} f(x) \, dx = F(b) - F(a) \), where \( F \) is any antiderivative of \( f \), that is, \( F' = f \).

**Total Change Theorem** The integral of a rate of change is the total change:

\[
\int_{a}^{b} F'(x) \, dx = F(b) - F(a)
\]

**Indefinite Integrals**

The notation \( \int f(x) \, dx \) is used for an antiderivative of \( f \) and is called an **indefinite integral**. Thus

\[
\int f(x) \, dx = F(x) \quad \text{means} \quad F'(x) = f(x)
\]

You should distinguish carefully between definite and indefinite integrals. A definite integral \( \int_{a}^{b} f(x) \, dx \) is a number, whereas an indefinite integral \( \int f(x) \, dx \) is a function.

**Evaluation Theorem** If \( f \) is continuous function on the interval \([a, b]\), then

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx \bigg|_{a}^{b} = F(b) - F(a)
\]

where \( F \) is any antiderivative of \( f \), that is \( F' = f \).
Table of Indefinite Integrals

\[ \int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx \]

\[ \int cf(x) \, dx = c \int f(x) \, dx \]

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad (n \neq -1) \]

\[ \int \frac{1}{x} \, dx = \ln |x| + C \]

\[ \int e^x \, dx = e^x + C \]

\[ \int a^x \, dx = \frac{a^x}{\ln a} + C \]

\[ \int \sin x \, dx = -\cos x + C \]

\[ \int \cos x \, dx = \sin x + C \]

\[ \int \sec^2 x \, dx = \tan x + C \]

\[ \int \csc^2 x \, dx = -\cot x + C \]

\[ \int \sec x \tan x \, dx = \sec x + C \]

\[ \int \csc x \cot x \, dx = -\csc x + C \]

\[ \int \frac{1}{x^2 + 1} \, dx = \arctan x + C \]

\[ \int \frac{1}{\sqrt{1 - x^2}} \, dx = \arcsin x + C \]