Lecture 4: Diffusion in a Wire

Introduction. In this lecture we consider heat conduction in a thin electrical wire, which is thermally insulated on its surface. The model of the temperature will have the form $\text{new} \ y = A \ \text{old} \ y + b$. In general, the matrix $A$ can be extremely large, but it will also have a special structure with many more zeros than nonzero components.

In this section we present a second model of heat transfer. In our first model we considered heat transfer via a discrete version of Newton's law of cooling which involves temperature as only a discrete function of time. That is, we assumed the mass was uniformly heated with respect to space. In this section we allow the temperature to be a function of both discrete time and discrete space.

The model for the diffusion of heat in space is based on empirical observations.

Discrete Fourier Heat Law:
   
   (a). heat flows from hot to cold,
   (b). the change in heat is proportional to the
cross sectional area,
change in time, and
(change in temperature)/(change in direction).

The last term is a good approximation provided the change in space is small. The proportionality constant, $K$, is called the thermal conductivity. The $K$ varies with the particular material and with the temperature. Here we will assume the temperature varies over a smaller range so that $K$ is approximately a constant.

Consider a thin wire so that there is diffusion in only one direction, $x$. The wire will have a current going through it so that there is a source of heat, $f$, which is from the electrical resistance of the wire. The $f$ has units of (heat)/(volume*time). We will assume the ends of the wire are kept at zero temperature, and the initial temperature is also zero. The goal is to be able to predict the temperature inside the wire for any future time and space location. Problems similar to this have important applications to devices used to measure wind speeds where the wind cools the wire enough so that the electrical resistance changes.
Model. In order to develop a model to predict the temperature \( u(x,t) \), we will discretize both space and time and let \( u(ih,k\Delta t) \) be approximated by \( u_i^k \) where \( \Delta t = T/\max_k \), \( h = L/n \) and \( L \) is the length of the wire. The model will have the general form

\[
\text{change in heat content} \approx (\text{heat from the source}) \\
+ (\text{heat from diffusion through the right end}) \\
+ (\text{heat from diffusion through the left end}).
\]

This is depicted in the figure below where the time step has not been indicated. For time we can choose either \( k\Delta t \) or \( (k+1) \Delta t \). Presently, we will choose \( k\Delta t \), and this will result in the matrix version of the first order finite difference method.

\[
\begin{array}{cccc}
  & u_{i-1} & u_i & u_{i+1} \\
\cdots & * & * & * & * & \cdots \\
\end{array}
\]

\[ h \\
A \]

\( h \) is the change in length  
\( A \) is the cross section area  
\( u_i \) is the approximate temperature.

**Figure:** Heat Diffusion in a Thin Wire

The heat diffusing in the right face (when \( (u_{i+1}^k - u_i^k)/h > 0 \)) is

\[
A \Delta t K (u_{i+1}^k - u_i^k)/h.
\]

The heat diffusing out the left face (when \( (u_i^k - u_{i-1}^k)/h > 0 \)) is

\[
A \Delta t K (u_i^k - u_{i-1}^k)/h.
\]

Therefore, the heat from diffusion is

\[
A \Delta t K (u_{i+1}^k - u_i^k)/h - A \Delta t K (u_i^k - u_{i-1}^k)/h.
\]

The heat from the source is

\[
A h \Delta t f.
\]

The heat content of the volume \( Ah \) at time \( k\Delta t \) is

\[
\rho c u_i^k Ah \text{ where } \rho \text{ is the density and } c \text{ is the specific heat.}
\]

By combining these we have the following approximation of the change in the heat content for the small volume \( Ah \):

\[
\rho c u_i^{k+1} Ah - \rho c u_i^k Ah = A h \Delta t f + A \Delta t K (u_{i+1}^k - u_i^k)/h - A \Delta t K (u_i^k - u_{i-1}^k)/h.
\]

Now, divide by \( \rho c Ah \), define \( \alpha = (K/\rho c) (\Delta t/h^2) \) and explicitly solve for \( u_i^{k+1} \).
Explicit Finite Difference Model of Heat Transfer.

\[ u_{i}^{k+1} = \Delta t/\rho c f + \alpha (u_{i+1}^{k} + u_{i-1}^{k}) + (1 - 2\alpha) u_{i}^{k} \]  where

\[ i = 1, ..., n-1, \]
\[ k = 0, ..., \text{max}k-1, \]
\[ u_{i}^{0} = 0 \text{ for } i = 1, ..., n-1 \text{ and } \]
\[ u_{0}^{k} = u_{n}^{k} = 0 \text{ for } k = 1, ..., \text{max}k. \]

Equation (2) is the initial temperature set equal to zero, and (3) is the temperature at the left and right ends set equal to zero.

Equation (1) may be put into the matrix version of the first order finite difference method. For example, if the wire is divided into four equal parts, then \( n = 4 \) and (1) may be written as three scalar equations, or as one 3D vector equation:

\[ u_{1}^{k+1} = \Delta t/\rho c f + \alpha (u_{2}^{k} + 0) + (1 - 2\alpha) u_{1}^{k} \]
\[ u_{2}^{k+1} = \Delta t/\rho c f + \alpha (u_{1}^{k} + u_{3}^{k}) + (1 - 2\alpha) u_{2}^{k} \]
\[ u_{3}^{k+1} = \Delta t/\rho c f + \alpha (0 + u_{2}^{k}) + (1 - 2\alpha) u_{3}^{k} \]

\[ u^{k+1} = A u^{k} + b \]

where

\[ u^{k} = \begin{bmatrix} u_{1}^{k} \\ u_{2}^{k} \\ u_{3}^{k} \end{bmatrix}, \quad b = (\Delta t / \rho c) f \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 - 2\alpha & \alpha & 0 \\ \alpha & 1 - 2\alpha & \alpha \\ 0 & \alpha & 1 - 2\alpha \end{bmatrix}. \]

An extremely important restriction on the time step \( \Delta t \) is required to make sure the algorithm is stable. For example, consider the case \( n = 2 \) where the above is the scalar equation, and we have the simplest first order finite difference model. Here \( a = 1 - 2\alpha \) and to avoid a blowup we must require \( a = 1 - 2\alpha > 0 \) and \( \alpha > 0 \). If \( n \) is larger than 2, this simple condition will imply that the matrix products \( A^{k} \) will converge to the zero matrix. This will make sure there are no blowups provided the source term \( f \) is bounded. The illustration of the stability condition, and an analysis will be presented later.

**Stability Condition for (1).**

\[ 1 - 2\alpha > 0 \text{ and } \alpha > 0 \]

**Method.** In order to compute all \( u_{i}^{k+1} \), which we will henceforth denote by \( u(i,k+1) \) with both \( i \) and \( k \) shifted up by one, we must use a nested loop where the \( i \) loop (space) is inside and the \( k \) loop (time) is the outer loop. This is illustrated in the following figure by
the dependency of \( u(i,k+1) \) on the three previously computed \( u(i-1,k) \), \( u(i,k) \) and \( u(i+1,k) \).

![Time-Space Grid](image)

**Figure: Time-Space Grid**

**Implementation.** We give a Matlab implementation for the explicit heat transfer algorithm for heat transfer in a thin wire. The Matlab code illustrates the stability condition on the discretization.

**Matlab Code (heat.m)**

```matlab
% Heat Diffusion in a Thin Insulated Wire
clear;
% Lenth of the Wire
L = 1.0;
% Final Time
T = 150.;
% Number of Time Steps
maxk = 30;
dt = T/maxk;
% Number of Space Steps
n = 10.;
dx = L/n;
b = dt/(dx*dx);
cond = .001;
spheat = 1.0;
rho = 1.;
a = cond/(spheat*rho);
```
d = a*b;
%  Initial Temperature
for i = 1:n+1
    x(i) = (i-1)*dx;
    u(i,1) = sin(pi*x(i));
end
%  Boundary Temperature
for k=1:maxk+1
    u(1,k) = 0.;
    u(n+1,k) = 0.;
    time(k) = (k-1)*dt;
end
%  Time Loop
for k=1:maxk
    %  Space Loop
    for i=2:n;
        u(i,k+1) = 1.]*dt/(spheat*rho)+(1-2*d)*u(i,k)
        + d*(u(i-1,k)+u(i+1,k));
    end
end
mesh(x,time,u')

Figure: Stable Computation of Temperature
The second calculation increases the time step from 5 to 6 so that the stability condition does not hold. Note that significant oscillations develop, which is not an attribute of heat diffusion!

**Figure:** Unstable Computation of Temperature

**Assessment.** The heat conduction in a thin wire has a number of approximations. Different mesh sizes in either the time or space variable will give different numerical results. However, if the stability conditions holds and the mesh sizes decrease, then the numerical computations will differ by smaller amounts.

In the scalar version of first order finite difference models the scheme was stable when $|a| < 1$. In this case, $y^{k+1}$ converged to the steady state solution $y = ay + b$. This is also true of the matrix version of (1) provided the stability condition is satisfied. In this case the real number $a$ will be replaced by the matrix $A$, and $A^k$ will converge to the zero matrix.
The above discrete model is derived via the discrete version of the Fourier heat law. It can be formulated as either a continuous model or as a discrete model. The discrete model is
\[ \rho c u_{i+1}^{k+1} Ah - \rho c u_i^k Ah = A h \Delta t f + A \Delta t K (u_{i+1}^k - u_i^k)/h - A \Delta t K (u_i^k - u_{i-1}^k)/h. \]
Divide by Ah \( \Delta t \)
\[ \rho c (u_{i+1}^{k+1} - u_i^k)/ \Delta t = f + (K(u_{i+1}^k - u_i^k)/h - K(u_i^k - u_{i-1}^k)/h)/h. \]
Approximate the first order partial derivatives
\[ \rho c u_t(ih, k\Delta t + \Delta t/2) = f + (Ku_x(ih + h/2, k\Delta t) - Ku_x(ih - h/2, k\Delta t))/h. \]
Approximate the second order partial derivative to get a continuous model
\[ \rho c u_t = f + (Ku_x)_x \]
For appropriate choices of time and space steps the discrete and continuous solutions should be close.

**Continuous 1D Model.**
\[ \rho c u_t - (Ku_x)_x = f \]
\[ u(x,0) = \text{given and} \]
\[ u(0,t), u(1,t) = \text{given} \]
\[ u(0), u(1) \]

The steady state solution is given by setting \( u_t = 0 \) so that the temperature depends only on space and \( -(Ku_x)_x = f \) and \( u(0), u(1) \) are given. For example, if \( K = .001, f = 1, u(0) = 0 \) and \( u(1) = 0 \), then \( u(x) \) must be a quadratic function of \( x \)
\[ u(x) = -1000 x^2/2 + a x + b. \]
The boundary conditions imply the choice of \( a \) and \( b \) so that
\[ u(x) = -500x(1 - x). \]

**Homework.**
1. Write a computer program and observe the consequences of not satisfying the stability condition.
2. Experiment with the mesh size and observe convergence as the mesh size decreases.
3. What happens to the discrete solution as \( k \) increases?