Lecture 9: Analysis of von Neumann Series

Introduction. In the applications to heat and mass transfer the discrete time dependent models have the form
\[ y^{k+1} = A y^k + b. \]
Under stability conditions on the time step the solution would “converge” to the solution of the steady state problem
\[ y = A y + b. \]
One condition that ensured this was when \( A^k \) “converged” to the zero matrix, the \( y^{k+1} \) converges to \( y \). We would like to be more precise about the term “converge”.

Definition. The infinity norm of the vector \( x \) is a real number
\[ \|x\| = \max_{i} |x_i| \text{ where } 1 \leq i \leq n. \]
The infinity norm of the square matrix \( A = [a_{ij}] \) is
\[ \|A\| = \max_{i,j} \sum_j |a_{ij}|. \]

Example. Let \( x = \begin{bmatrix} -1 \\ 6 \\ -9 \end{bmatrix} \) and \( A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 3 & 1 \\ 3 & 0 & 5 \end{bmatrix} \).
\[ \|x\| = \max \{1,6,9\} = 9 \text{ and } \|A\| = \max \{8,5,8\} = 8. \]

Basic Properties.
1. \( \|x\| \geq 0, \text{ and } \|x\| = 0 \text{ if and only if } x = 0. \)
2. \( \|x + y\| \leq \|x\| + \|y\| \)
3. \( \|\alpha x\| = |\alpha| \|x\| \text{ where } \alpha \text{ is real.} \)
4. \( \|Ax\| \leq \|A\| \|x\| \)
5. \( \|AB\| \leq \|A\| \|B\| \).

The proofs of the first three basic properties are a consequence of the analogous properties of the absolute value. Real valued functions that satisfy 1-3 are called norms.
Proof of 4. In order to prove the fourth property, apply the definition of infinity norm to the matrix-vector product $Ax$

$$\|Ax\| = \max \left| \sum_j a_j x_j \right| \leq \max_i \sum_j |a_j| \left| x_j \right| \leq (\max_i \sum_j |a_j|) \cdot (\max_j \left| x_j \right|).$$

Proof of 5.

$$\|AB\| = \max_i \left| \sum_j \sum_k a_{ik} b_{kj} \right|$$

$$\leq \max_i \sum_j \sum_k |a_{ik}| |b_{kj}|$$

$$= \max_i \sum_k |a_{ik}| \sum_j |b_{kj}|$$

$$\leq (\max_i \sum_k |a_{ik}|)(\max_j \sum_j |b_{kj}|)$$

$$= \|A\| \|B\|$$

Definition. Let $y^k$ and $y$ be vectors. $y^k$ converges to $y$ if and only if each component of $y^k$ converge to $y_i$. This is equivalent to

$$\|y^k - y\| = \max_i |y^k_i - y_i|$$

converges to zero.

Like the geometric series of single numbers the iterative scheme

$$y^{k+1} = Ay^k + b$$

can be expressed as a summation via recursion.

$$y^{k+1} = Ay^k + b$$

$$= A(Ay^{k-1} + b) + b$$

$$= A^2y^{k-1} + Ab + b$$

$$= A^2(Ay^{k-2} + b) + Ab + b$$

$$= A^3y^{k-2} + (A^2 + A^1 + I)b$$

$$\vdots$$

$$= A^{k+1}y^0 + (A^k + \ldots + I)b.$$ (1)
**Definition.** The summation \( I + \ldots + A^k \) and the series \( I + \ldots + A^k + \ldots \) are generalizations of the geometric series, and they are often referred to as the von Neumann series.

Previously, we showed if \( A^k \) converges to the zero matrix, then \( y^{k+1} = A^k y + b \) must converge to the solution of \( y = A y + b \), which is also a solution of \( (I - A)y = b \). If \( I - A \) has an inverse, this and line (1) suggest that the von Neumann series must converge to the inverse of \( I - A \). If the norm of \( A \) is less than 1, then all this is true.

**Theorem.** Consider the scheme \( y^{k+1} = A^k y + b \) and let \( y \) satisfy \( y = A^k y + b \). If the norm of \( A \) is less than one, then

1. \( y^{k+1} = A^k y + b \) converges to \( y = A y + b \),
2. \( I - A \) has an inverse and
3. \( I + \ldots + A^k \) converges to the inverse of \( I - A \).

**Proof of 1.** \( y^{k+1} - y = A^k y + b - y \). Apply property 4 of the norm as follows:

\[
\| y^{k+1} - y \| = \| A^k (y - y) \| \\
\leq \| A \| \| y - y \| \\
\leq \| A \| \| A \| \| y^{k-1} - y \| \\
= \| A \|^2 \| y^{k-1} - y \| \\
\vdots \\
\leq \| A \|^k \| y^0 - y \|.
\]

Because the norm of \( A \) is less than one, the right side must go to zero.
This forces the norm of the error to go to zero.

**Proof of 2.** This requires some additional information from matrix algebra: a square matrix \( B \) has an inverse matrix if and only if \( Bx = 0 \) implies \( x = 0 \). Let \( B = I - A \) and suppose \( Bx = 0 \). Then \( 0 = Bx = (I - A)x = Ix - Ax \) so that \( x = Ax \). If \( x \neq 0 \), then by property 1 \( \| x \| \neq 0 \). Apply the norm to both sides of \( x = Ax \) and use property 4 to get \( \| x \| = \| Ax \| \leq \| A \| \| x \| \).
Now divide by the norm of \( x \) to get a contradiction to the assumption that the norm of \( A \) is strictly less than one.
Proof of 3. By the associative and distributive properties of matrices we have
\[
(I - A)(I + A + \cdots + A^k) = I(I + A + \cdots + A^k) - A(I + A + \cdots + A^k) \\
= I - A^{k+1}.
\]
Multiply both sides by the inverse of \(I - A\) to get
\[
(I + A + \cdots + A^k) = (I - A)^{-1}(I - A^{k+1}) \\
= (I - A)^{-1}I - (I - A)^{-1}A^{k+1} \\
(1 + A + \cdots + A^k) = -(I - A)^{-1}A^{k+1}.
\]
Apply property 5 of the norm to get
\[
\left\| (I + A + \cdots + A^k) - (I - A)^{-1} \right\| = \left\| - (I - A)^{-1}A^{k+1} \right\| \\
\leq \left\| -(I - A)^{-1} \right\| \left\| A^{k+1} \right\| \\
\leq \left\| -(I - A)^{-1} \right\| \left\| A \right\|^{k+1}.
\]
Since the norm is less than one the right side must go to zero. Thus, the series must converge to the inverse of \(I - A\).

Application to the Cooling Wire. Consider a cooling wire with the heat loss through the lateral surface. It will be assumed to be directly proportional to the change in time, the lateral surface area and to the difference in the surrounding temperature and the temperature in the wire. Let \(c_{\text{sur}}\) be the proportionality constant, which measures insulation. Let \(r\) be the radius of the wire so that the lateral surface area of a small wire segment is \(2\pi rh\). If \(u_{\text{sur}}\) is the surrounding temperature of the wire, then the heat loss through the small lateral area is \(c_{\text{sur}} \Delta t 2\pi rh (u_{\text{sur}} - u_i)\). Heat loss or gain from a source such as electrical current and from left and right diffusion will remain the same as in the previous lecture. Let \(\alpha = (K/\rho c)(\Delta t/\Delta x^2)\) and \(\Delta x = L/(n+1)\) and combine the above.

Explicit Finite Difference Model of Heat Transfer.
\[
u_i^{k+1} = \Delta t/\rho c (f + c_{\text{sur}} (2/r) u_{\text{sur}}) + \alpha(u_{i+1}^k + u_{i-1}^k) + \\
(1 - 2\alpha - (\Delta t/\rho c) c_{\text{sur}} (2/r)) u_i^k \quad \text{where} \quad i = 1, \ldots, n-1, \quad k = 0, \ldots, \text{maxk}, \quad u_i^0 = 0 \quad \text{for} \ i = 1, \ldots, n-1 \quad \text{and} \quad u_0^k = u_n^k = 0 \quad \text{for} \ k = 1, \ldots, \text{maxk}. \quad (2)
\]
Equation (3) is the initial temperature set equal to zero, and (4) is the temperature at the left and right ends set equal to zero.

Equation (2) may be put into the matrix version of the first order finite difference method. For example, if the wire is divided into four equal parts, then \( n = 4 \) and (1) may be written as a 3D vector equation

\[
\mathbf{u}^{k+1} = \mathbf{A} \mathbf{u}^k + \mathbf{b}
\]

where

\[
\mathbf{u}^k = \begin{bmatrix} u_1^k \\ u_2^k \\ u_3^k \end{bmatrix}, \quad \mathbf{b} = (\Delta t / \rho c) \mathbf{F} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 - 2\alpha - d & \alpha & \alpha \\ \alpha & 1 - 2\alpha - d & \alpha \\ \alpha & \alpha & 1 - 2\alpha - d \end{bmatrix}
\]

with

\[
\mathbf{F} = f + c_{\text{sur}} (2 / r) u_{\text{sur}} \quad \text{and} \quad d = (\Delta t / \rho c) c_{\text{sur}} (2 / r).
\]

**Stability Condition for (2).**

\[
1 - 2\alpha - (\Delta t / \rho c) c_{\text{sur}} (2/r) > 0 \quad \text{and} \quad \alpha > 0.
\]

The norm of the above 3x3 matrix under the stability condition is

\[
\max \{ |1 - 2\alpha - d| + |\alpha| + |0|, |\alpha| + |1 - 2\alpha - d| + |\alpha|, |1 - 2\alpha - d| + |\alpha| \}
\]

\[
= \max \{ 1 - 2\alpha - d + \alpha, \alpha + 1 - 2\alpha - d + \alpha, 1 - 2\alpha - d + \alpha \}
\]

\[
= \max \{ 1 - \alpha - d, 1 - d, 1 - \alpha - d \}
\]

\[
= 1 - d < 1.
\]

**Application to Pollutant in a Stream.** The model will have the general form

change in amount \( \approx \) (amount entering from upstream) - (amount leaving to downstream) - (amount decaying in a time interval).

In lecture 7 we formulated the following discrete model for the approximation of the pollutant.

**Explicit Finite Difference Model of Flow and Decay of a Pollutant.**

\[
u_i^{k+1} = \text{vel} (\Delta t / \Delta x) u_{i-1}^k + (1 - \text{vel} (\Delta t / \Delta x) - \Delta t \text{dec}) u_i^k \quad \text{where} \quad (5)
\]

\[
i = 1,...,n-1,
\]
\[ k = 0, \ldots, \text{max}_k - 1, \]
\[ u_i^0 = \text{given for } i = 1, \ldots, n-1 \text{ and} \]
\[ u_0^k = \text{given for } k = 1, \ldots, \text{max}_k. \]

Equation (6) is the initial concentration, and (7) is the concentration far upstream.

Equation (5) may be put into the matrix version of the first order finite difference method. For example, if the stream is divided into four equal parts, then \( n = 4 \) and (1) may be written as 3D vector equation

\[
\begin{bmatrix}
    u_1^{k+1} \\
    u_2^{k+1} \\
    u_3^{k+1}
\end{bmatrix} = \begin{bmatrix}
    c & u_1^k \\
    d & c & u_2^k \\
    d & d & c
\end{bmatrix}
\begin{bmatrix}
    du_0^k \\
    0 \\
    0
\end{bmatrix}
\]

where \( d = \text{vel} (\Delta t / \Delta x) \) and \( c = 1 - d - \text{dec} \Delta t \).

**Stability Condition for (5).**

\[
1 - \text{vel} (\Delta t/\Delta x) - \text{dec} \Delta t \text{ and vel, dec} > 0.
\]

When the stability condition holds, the norm of the 3x3 matrix is given by

\[
\max \{|c| + |0| + |0|, |d| + |c| + |0|, |d| + |c|\}
\]

\[
= \max \{1 - d - \text{dec} \Delta t, d + 1 - d - \text{dec} \Delta t, d + 1 - d - \text{dec} \Delta t\}
\]

\[
= 1 - \text{dec} \Delta t < 1.
\]

**Homework.**

1. Find the norms of the following:

\[
x = \begin{bmatrix}
    1 \\
    -7 \\
    0 \\
    3
\end{bmatrix}
\]

and \( A = \begin{bmatrix}
    4 & -5 & 3 \\
    0 & 10 & -1 \\
    11 & 2 & 4
\end{bmatrix} \).

2. Verify properties 1-3 of the norm.

3. Consider the application to cooling. Let \( n = 5 \). Find the matrix and determine when its norm will be less than one.

4. Consider the application to pollution of stream. Let \( n = 5 \). Find the matrix and determine when its norm will be less than one.