Lecture 8: Pollutant Transfer in a Lake

Introduction. In the previous lecture a pollutant was modeled as it was dispersed down a shallow stream. In this case the concentration was a function of time and just one space variable, x. If the pollutant was dispersed in a shallow lake, then the concentration would be a function of the position on the lake's surface, and so, the concentration would be a function of time and two space variables, x and y.

Model. Let $u(x,y,t)$ be the concentration of a pollutant. Suppose it is decaying at a rate equal to $\text{dec}$ units per time, and it is being dispersed to other parts of the lake by a known wind with constant velocity vector equal to $(v_1, v_2)$. 

![Diagram of a lake with pollutant source and wind velocity vectors](image-url)
Following the derivations in previous lecture, but now consider both directions, we obtain the continuous and discrete model. We have assumed both the velocity components are nonnegative so that the concentrations on the upwind side (west and south) sides must be given. The amount entering the left face of the rectangular region is given by

\[(T \Delta y) (v_1 \Delta t) u_{i-1,j}^k\]

where

\[T\] is the depth, the volume entering is \((T \Delta y) (v_1 \Delta t)\), and the concentration is \(u_{i-1,j}^k\). The total change in the amount inside this rectangular region over a time interval \(\Delta t\) is

\[
(T \Delta x \Delta y) (u_{ij}^{k+1} - u_{ij}^k) = (T \Delta y) (v_1 \Delta t) u_{i-1,j}^k - (T \Delta y) (v_1 \Delta t) u_{ij}^k
+ (T \Delta x) (v_2 \Delta t) u_{ij-1}^k - (T \Delta x) (v_2 \Delta t) u_{ij}^k
\Delta t \text{ dec} (T \Delta x \Delta y) u_{ij}^k.
\]

In the partial differential equation in the continuous 2D model the term \(-v_2 u_y\) models the amount of the pollutant entering and leaving the top and bottom of a rectangle region.

**Continuous 2D Pollutant Model.**

\[u_t = -v_1 u_x - v_2 u_y - \text{dec} u, \quad (1.1)\]
\[u(x,y,0) = \text{given} \quad (1.2)\]
\[u(x,y,t) = \text{given on the upwind boundary of the lake}. \quad (1.3)\]

**Explicit Finite Difference 2D Pollutant Model.**

\[u_{ij}^{k+1} = v_1 (\Delta t/\Delta x) u_{i-1,j}^k + v_2 (\Delta t/\Delta y) u_{ij-1}^k +
(1 - v_1 (\Delta t/\Delta x) - v_2 (\Delta t/\Delta y) - \Delta t \text{ dec})u_{ij}^k \quad (2.1)\]
\[u_{ij}^0 = \text{given} \quad (2.2)\]
\[u_{0j} \text{ and } u_{i0} = \text{given}. \quad (2.3)\]

**Stability Condition.**

\[1 - v_1 (\Delta t/\Delta x) - v_2 (\Delta t/\Delta y) - \Delta t \text{ dec} > 0.\]

**Method.** In order to execute the discrete model, there must be three nested loops. The outer loop must be the time loop (uses \(k\)), and the two inner loops are for the space grid (uses \(i\) and \(j\)). The order the \(i\) (\(x\) direction) and the \(j\) (\(y\) direction) is not important so long as they are both within the \(k\) (time loop).
Implementation. The following Matlab code simulates a large spill of a pollutant along the southwest boundary of a lake. The source of the spill is controlled after 25 time steps so that the pollutant "cloud" moves across the lake as depicted by the contour plots for different times. The Matlab code `flow2d` generates the 3D array of the concentrations, and the Matlab code in `mov2d.m` generates a sequence of contour plots.

Matlab Codes for 2D Pollutant Flow (flow2d.m and mov2d.m)

```matlab
% Flow2d Model in File flow2d.m
clear;
L = 1.0;
W = 4.0;
T = 10.;
K = 200;
dt = T/K;
n = 10.;
dx = L/n;
m = 20.;
dy = W/m;
velx = .1;
vely = .4;
decay = .1;
for i = 1:n+1
    x(i) = (i-1)*dx;
    for j = 1:m+1
        y(j) = (j-1)*dy;
        u(i,j,1) = 0.;
    end
end
for k=1:K+1
    time(k) = (k-1)*dt;
    for j=1:m+1
        u(1,j,k) = .0;
    end
    for i=1:n+1
        u(i,1,k) = (i<=(n/2+1))*(k<26)*5.0*sin(pi*x(i)*2) + (i>(n/2+1))*1;
    end
end
```
for k=1:K
    for i=2:n+1;
        for j=2:m+1;
            u(i,j,k+1) = (1 - velx*dt/dx - vely*dt/dy - decay*dt)*u(i,j,k) + velx*dt/dx*u(i-1,j,k) + vely*dt/dy*u(i,j-1,k);
        end
    end
end
mesh(x,y,u(:,:,K)');

% Code in mov2d.m
flow2d;
lim = [0 1 0 4];
for k=1:5:200
    contour(x,y,u(:,:,k)');
    title ('concentration versus space at different times')
    axis(lim);
    k = waitforbuttonpress;
end
Assessment. Here we have assumed the only mechanism for transfer of the pollutant is via movement of the lake water. Particles may also move in a similar manner as the flow of heat moving from regions of high concentrations to regions of lower concentrations. Groundwater pollution is another related model. Here the velocity of the flow will be proportional to the gradient of the pressure, as is the rate of heat flow is proportional to the gradient of the temperature. Because the proportionality “constant” depends on the underground soil, it may not be explicitly known or even a constant!

Homework.

1. Experiment with the parameters in the Matlab code for a pollutant in a shallow lake. Observe stability as a function of wind speeds.
2. How would you modify the shallow water lake model to include movement in the vertical direction?
3. How would you modify the shallow water lake model to include “diffusion” of the pollutant?