**Lecture 5: Analysis of Stability**

**Introduction.** In this lecture we continue to consider heat conduction in a thin electrical wire which is thermally insulated on its surface. The time dependent model of the temperature will have the form $\text{new}_y = A \text{old}_y + b$. In general, the matrix $A$ can be extremely large, but it will also have a special structure with many more zeros than nonzero components. As time marches one can expect the temperature to reach a steady state so that $y = A y + b$.

In the previous lecture we formulated the discrete model based on the Fourier law for heat diffusion in a thin wire.

**Explicit Finite Difference Model of Heat Transfer.**

$$u_i^{k+1} = \Delta t/\rho c f + \alpha (u_{i+1}^k + u_{i-1}^k) + (1 - 2\alpha) u_i^k \quad \text{where}$$

$$i = 1, \ldots, n-1,$$

$$k = 0, \ldots, \text{maxk}-1,$$

$$u_i^0 = 0 \quad \text{for} \quad i = 1, \ldots, n-1 \quad \text{and}$$

$$u_0^k = u_n^k = 0 \quad \text{for} \quad k = 1, \ldots, \text{maxk}. \quad (2)$$

Equation (2) is the initial temperature set equal to zero, and (3) is the temperature at the left and right ends set equal to zero.

Equation (1) may be put into the matrix version of the first order finite difference method. For example, if the wire is divided into four equal parts, then $n = 4$ and (1) may be written as either as three scalar equations, or as one 3D vector equation.

$$u_1^{k+1} = \Delta t/\rho c f + \alpha (u_2^k + 0) + (1 - 2\alpha) u_1^k$$

$$u_2^{k+1} = \Delta t/\rho c f + \alpha (u_3^k + u_1^k) + (1 - 2\alpha) u_2^k$$

$$u_3^{k+1} = \Delta t/\rho c f + \alpha (0 + u_2^k) + (1 - 2\alpha) u_3^k$$

$$u_4^{k+1} = A u^k + b \quad \text{where}$$

$$u^k = \begin{bmatrix} u_1^k \\ u_2^k \\ u_3^k \end{bmatrix}, \quad b = (\Delta t / \rho c)f \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 - 2\alpha & \alpha \\ \alpha & 1 - 2\alpha & \alpha \\ \alpha & \alpha & 1 - 2\alpha \end{bmatrix}. \quad (4)$$

An extremely important restriction on the time step $\Delta t$ is required to make sure the algorithm is stable. Here $a = 1 - 2\alpha > 0$ is needed to avoid a blowup of the numerical solution. This simple condition implies that the matrix products $A^k$ will converge to the
zero matrix. This will make sure there are no blowups provided the source term $f$ is bounded.

**Stability Condition for (1).**

$$1 - 2\alpha > 0 \text{ and } \alpha > 0.$$ 

**Matrix Products.** In order to carefully study stability, we will need some basic facts about matrices. The simplest matrices (also called arrays) are row and column vectors. For example, $A = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$ is a 3\times1 row vector and $B = \begin{bmatrix} 1 \\ 8 \\ -1 \\ 10 \end{bmatrix}$ is a 4\times1 column vector.

If the row and column vectors have the same number of components, then we define their product to be a single number given by the sum of the products of the components. For example,

$$A = \begin{bmatrix} 1 & 4 & 0 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 \\ 7 \\ 9 \\ 10 \end{bmatrix}$$

$$AB = 1*2 + 4*7 + 0*9 + 8*10 = 110.$$ 

**Definition.** Let $A = [a_k]^T$ denote a 1\times n row vector where $k$ varies from 1 to $n$.

Let $B = [b_k]$ denote a n\times 1 column vector where $k$ varies from 1 to $n$.

Define the row-column matrix product of $A$ and $B$ to be a single number

$$AB = \sum_{k=1}^{n} a_k b_k.$$ 

If a matrix has $m$ rows and $n$ columns, then we may denote it by

$$A = [a_{ik}] \text{ where } i \text{ varies from } 1 \text{ to } m \text{ and } j \text{ varies from } 1 \text{ to } n.$$ 

In this case $A$ is called an $m \times n$ matrix. If $A$ is $m \times n$ and $B = [b_{kj}]$ is $n \times p$ so that $j$ varies from 1 to $p$, then we can define a matrix product $AB$ to be a $m \times p$ matrix whose $ij$ components are row $i$ of $A$ times column $j$ of $B$. For example,
\[ A = \begin{bmatrix} 1 & 6 & 5 \\ 7 & 8 & 9 \end{bmatrix} \]
\[ B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \]
\[ AB = \begin{bmatrix} 1*5 + 6*5 + 5*9 & 88 & 100 & 112 \\ 7*1 + 8*5 + 9*9 & 152 & 176 & 200 \end{bmatrix} = \begin{bmatrix} 76 & 88 & 100 & 112 \\ 128 & 152 & 176 & 200 \end{bmatrix}. \]

It is important to note that the product BA may not be defined even if AB is defined. In general, BA and AB are not equal for matrices A and B!!!!

**Definition.** Let \( A = [ a_{ik} ] \) be mxn and \( B = [ b_{kj} ] \) be nxp. The **matrix product** \( AB \) is defined to be a mxp matrix whose \( ij \) component is the product of row \( I \) of \( A \) and column \( j \) of \( B \), that is,
\[ AB = \sum_{k=1}^{m} a_{ik} b_{kj}. \]

**Definition.** Let \( B \) and \( D \) both be nxp matrices. The **matrix sum** of \( B \) and \( D \) is defined to be a third nxp matrix whose \( kj \) components are the sum of the \( kj \) components of \( B \) and \( D \), that is,
\[ B + D = [ b_{kj} + d_{kj} ]. \]

**Proposition.** Let \( A \) be mxn, \( B \) and \( D \) be nxp and \( C \) be pxq matrices. Then the following are true:
1. \( A(B+D) = AB + AD \), distributive property
2. \( A(BC) = (AB)C \), associative property.
Proof of 1.

\[ B + D = [b_{kj} + d_{kj}] \]
\[ A(B + D) = \left[ \sum_{k=1}^{m} a_{ik} \right] (b_{kj} + d_{kj}) \]
\[ = \left[ \sum_{k=1}^{m} (a_{ik} b_{kj}) + \sum_{k=1}^{m} (a_{ik} d_{kj}) \right] \quad \text{by the distributive property for real numbers} \]
\[ = \left[ \sum_{k=1}^{m} (a_{ik} b_{kj}) \right] + \left[ \sum_{k=1}^{m} (a_{ik} d_{kj}) \right] \quad \text{by the definition of the summation operation} \]
\[ = AB + AD \quad \text{by the definition of matrix addition} \]

Proof of 2.

\[ A(BC) = \left[ \sum_{k=1}^{m} a_{ik} \right] \left( \sum_{j=1}^{p} b_{kj} c_{jl} \right) \]
\[ = \left[ \sum_{k=1}^{m} \sum_{j=1}^{p} a_{ik} \right] (b_{kj} c_{jl}) \quad \text{by definition of the summation symbol} \]
\[ = \left[ \sum_{k=1}^{m} \sum_{j=1}^{p} (a_{ik} b_{kj}) \right] c_{jl} \quad \text{by the associative property for real numbers} \]
\[ = \left[ \sum_{j=1}^{p} \sum_{k=1}^{m} (a_{ik} b_{kj}) \right] c_{jl} \quad \text{by commutative property of addition} \]
\[ = (AB)C \quad \text{by the definition of matrix products} \]

Analysis of Stability. The heat conduction in a thin wire has a number of approximations. Different mesh sizes in either the time or space variable will give different numerical results. However, if the stability conditions holds and the mesh sizes decreases, then the numerical computations will differ by smaller amounts.

In the scalar version of first order finite difference models the scheme was stable when \( a = 1 - 2\alpha > 0 \). In this case, \( y^{k+1} \) converged to the steady state solution \( y = a y + b \). This is also true of the matrix version of (1) provided the stability condition is
satisfied. In this case the real number $a$ will be replaced by the matrix $A$, and $A^k$ will converge to the zero matrix.

**Steady State Theorem.** Consider the matrix version of the first order finite difference equation

$$y^{k+1} = A y^k + b$$

where $A$ is a square matrix.

If $A^k$ converges to the zero matrix and $y = Ay + b$, then, regardless of the initial choice for $y^0$, $y^k$ converges to $y$.

**Proof.** Subtract $y^{k+1} = A y^k + b$ and $y = A y^k + b$ and use the properties of matrix products to get

$$y^{k+1} - y = (A y^k + b) - (A y + b)$$

$$= A(y^k - y)$$

(by the distributive property)

$$= A(A(y^{k-1} - y))$$

(by recursion)

$$= A^2 (y^{k-1} - y)$$

(by the associative property)

$$\vdots$$

$$= A^k (y^0 - y).$$

Since $A^k$ converges to the zero matrix, the column vectors $y^{k+1} - y$ must converge to zero.

The assumption that $A^k$ must converge to the zero matrix seems difficult to verify. However, by using “matrix norms” we can formulate an easy test, which ensures the stability conditions on the time step will imply this assumption is true.

**Homework.**

1. Consider the 3x3 $A$ matrix in (4). Observe $A^k$ for different values of $\alpha$ so that the stability condition either does or does not hold.
2. Vary (4) so that there are $n = 5$ equal parts of the wire.
3. Find 2x2 matrices, $A$ and $B$, such that $AB$ is not equal to $BA$. 