i) Show that
\[ x^3 \frac{d^3 y}{dx^3} = D(D - 1)(D - 2)y \] (14)
and
\[ x^4 \frac{d^4 y}{dx^4} = D(D - 1)(D - 2)(D - 3)y. \] (15)
(Exercises 18 through 21 require Section 3.10.) Use formulas (12)–(15) above to change the following

---

### 3.12 Undetermined Coefficients (Second-Order)

As pointed out in Section 3.4, the general solution of
\[ ay''(x) + by'(x) + cy(x) = f(x). \] (1)
where \( a, b, c \) are constants and \( a \neq 0 \), may be written in the form \( y = y_p + y_h \), where \( y_h \) is the general solution of the associated homogeneous equation \( ay'' + by' + cy = 0 \) and \( y_p \) is a particular solution of \( ay'' + by' + cy = f \). The homogeneous equation was solved in Section 3.8. Two methods for finding the particular solution \( y_p \) will be given in this chapter. The method of undetermined coefficients will be developed in this section. The method of variation of parameters will be discussed in Sections 3.14 and 3.15.

The method of undetermined coefficients can be presented in (at least) two ways. While mathematically equivalent, the approaches appear different. In this section we present an approach that utilizes only our second-order results. In Section 3.13.2 an alternative approach is presented that requires operator notation (Section 3.6) and \( n \)-th-order homogeneous equations (Section 3.10).

If we think of the differential equation (1) as modeling a physical system, say a circuit, then \( f \) often stands for the input or outside influence, and the solution \( y \) is the response. The response is often similar to the input function.

**Example 3.12.1**

Find a particular solution of
\[ y'' + y = 3e^{-x}. \] (2)

Here \( f(x) = 3e^{-x} \) is an exponential. Let us see whether there is a solution of (2) of the form \( y_p = Ae^{-x} \) for some constant \( A \). Substituting this form for \( y_p \) into (2) for \( y \) gives
\[ (Ae^{-x})'' + Ae^{-x} = 3e^{-x} \]
or \( 2A = 3 \). Thus \( y_p = \frac{3}{2}e^{-x} \) is a particular solution of (2). ■
Example 3.12.2

Let us now try to find a particular solution of

\[ y'' - y = 3e^{-x}. \]  

(3)

Like Example 3.12.1, we have \( f(x) = 3e^{-x} \), so that we might try \( y_p = Ae^{-x} \) again. Substituting this form for \( y_p \) into (3), we get

\[ (Ae^{-x})'' - (Ae^{-x}) = 3e^{-x}, \]

or, after simplification, \( 0 = 3 \), which is impossible. Thus there is no particular solution of the form \( Ae^{-x} \). Note that \( e^{-x} \) is also a solution of the associated homogeneous equation \( y'' - y = 0 \). Letting \( y_1 = e^{-x} \) and using reduction of order (Section 3.7), we find, after some calculation, that

\[ y = c_1 e^{-x} + c_2 e^x - \frac{3}{2} x e^{-x} \]

is the general solution of (3) and \(-\frac{3}{2} x e^{-x}\) is a particular solution. \( \blacksquare \)

Examples 1 and 2 are typical of how the method of undetermined coefficients works. The \( f \) in (1) determines the form of a particular solution up to certain powers of \( x \). The powers of \( x \) are determined by the solutions of the associated homogeneous equation, or equivalently, by the roots of the characteristic equation. Once the form for a particular solution is determined, the actual particular solution is determined by substituting the form into the original differential equation and solving for the constants.

We shall give several important special cases of the method of undetermined coefficients and work several examples. These special cases are summarized in Table 3.12.1. There is a pattern to these special cases which the reader should look for. Finally, the general procedure will be given.

Special Case 1

If \( f(x) \) in (1) is an \( m \)-th-degree polynomial, then \( y_p \) is of the form \( x^k(A_0 + A_1 x + \cdots + A_m x^m) \), where \( k \) is the multiplicity of 0 as a root of the characteristic polynomial.

Example 3.12.3

\textbf{Solution}

Find the general solution of

\[ y'' - 3y' = 2x^2 + 1. \]  

(4)

The characteristic polynomial \( r^2 - 3r \) has roots \( r_1 = 0, r_2 = 3 \). Thus \( y_h = c_1 e^{0x} + c_2 e^{3x} = c_1 + c_2 e^{3x} \). Here \( f(x) = 2x^2 + 1 \) is a second-degree polynomial and 0 is a root of multiplicity 1, so that \( k = 1 \). Thus, by Special Case 1, \( y_p \) has the form \( x(A_0 + A_1 x + A_2 x^2) \). Substituting this form into (4) gives
Table 3.12.1 Special Cases

<table>
<thead>
<tr>
<th>$f$ includes summands of form</th>
<th>$y_p$ then includes</th>
<th>$k$ is the multiplicity of the root</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $p(x)$, an $m$th-degree polynomial</td>
<td>$x^4(A_0 + A_1x + \cdots + A_m x^m)$</td>
<td>0</td>
</tr>
<tr>
<td>2. $Ee^{ax}$</td>
<td>$x^k A e^{ax}$</td>
<td>2</td>
</tr>
<tr>
<td>3. $p(x)e^{ax}$</td>
<td>$x^4(A_0 + A_1x + \cdots + A_m x^m)e^{ax}$</td>
<td>2</td>
</tr>
<tr>
<td>$p(x)$ an $m$th-degree polynomial</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. $E_1 \cos \beta x + E_2 \sin \beta x$</td>
<td>$x^4(A_0 \cos \beta x + B_0 \sin \beta x)$</td>
<td>$\beta i$</td>
</tr>
<tr>
<td>5. $p(x) \cos \beta x + q(x) \sin \beta x$</td>
<td>$x^4(A_0 + A_1x + \cdots + A_s x^s) \cos \beta x + \sin \beta x$</td>
<td>$\beta i$</td>
</tr>
<tr>
<td>where $p(x)$ is an $m$th-degree polynomial and $q(x)$ is an $n$th-degree polynomial</td>
<td>$x^4(B_0 + B_1x + \cdots + B_s x^s) \sin \beta x$; $s = \text{larger of } m, n$</td>
<td></td>
</tr>
<tr>
<td>6. $E_1 e^{ax} \cos \beta x + E_2 e^{ax} \sin \beta x$</td>
<td>$x^4 e^{ax}(A_0 \cos \beta x + B_0 \sin \beta x)$</td>
<td>$x + \beta i$</td>
</tr>
</tbody>
</table>

\[(A_0 x + A_1 x^2 + A_2 x^3) - 3(A_0 x + A_1 x^2 + A_2 x^3)' = 2x^2 + 1\]

or

\[2A_1 + 6A_2 x - 3A_0 - 6A_1 x - 9A_2 x^2 = 2x^2 + 1.\]

Equating coefficients of like powers of $x$,

1: \[2A_1 - 3A_0 = 1,\]
2: \[6A_2 - 6A_1 = 0,\]
3: \[-9A_2 = 2,\]

we get $A_2 = -\frac{2}{3}$, $A_1 = -\frac{1}{3}$, and $A_0 = -\frac{4}{3}$. Thus the particular solution is $y_p = -\frac{4}{3}x - \frac{1}{3}x^2 - \frac{1}{3}x^3$ and the general solution is

\[y = y_p + y_h = -\frac{1}{3}x - \frac{1}{3}x^2 - \frac{1}{3}x^3 + c_1 + c_2 e^{3x}.\]

**Special Case 2**

If $f(x)$ is a constant times $e^{ax}$, then $y_p$ has the form $A x^k e^{ax}$, where $k$ is the multiplicity of $x$ as a root of the characteristic polynomial.

**Example 3.12.4**

Find the general solution of

\[y'' - 5y' + 6y = 4e^{2x}.\]

**Solution**

First we solve the homogeneous equation $y'' - 5y' + 6y = 0$. The characteristic equation is $r^2 - 5r + 6 = (r - 2)(r - 3) = 0$ and it has roots $r = 2, 3$. Therefore, $y_h = c_1 e^{2x} + c_2 e^{3x}$. According to Special Case 2, $y_p$ is of the form $y = x^k A e^{2x}$ (here $x = 2$) and $k = 1$, since 2 is a root of the characteristic equation of mult...
3.12 Undetermined Coefficients (Second-Order)

...form for \( y_p \) into the original equation (5), we obtain

\[
(Ax e^{2x})'' - 5(Ax e^{2x})' + 6(Ax e^{2x}) = 4e^{2x}
\]
or

\[
4Axe^{2x} + 4Ae^{2x} - 5(Ae^{2x} + 2Axe^{2x}) + 6Axe^{2x} = 4e^{2x}.
\]

Then \(-Ae^{2x} = 4e^{2x}\) and \(A = -4\). Thus \(y_p = -4xe^{2x}\) is a particular solution of (5) and

\[y = y_p + y_h = -4xe^{2x} + c_1e^{2x} + c_2e^{3x}\]
is the general solution of (5). #

Example 3.12.5

Find the general solution of

\[
y'' - 5y' - 6y = 4e^{2x}.
\]  \hspace{1cm} (6)

First we solve the associated homogeneous equation \(y'' - 5y' - 6y = 0\). The characteristic polynomial is \(r^2 - 5r - 6 = (r - 6)(r + 1)\) and has roots 6, -1. Thus \(y_h = c_1e^{6x} + c_2e^{-x}\). According to Special Case 2, \(y_p\) is of the form \(xe^{ax}\) (here \(a = 2\)) and \(k = 0\) since 2 is not a root of the characteristic equation. Thus \(y_p = Ae^{2x}\) for some constant \(A\). Substitute this form for \(y_p\) into the original equation (6)

\[
(Ae^{2x})'' - 5(Ae^{2x})' - 6(Ae^{2x}) = 4e^{2x},
\]
and simplify

\[
4Ae^{2x} - 10Ae^{2x} - 6Ae^{2x} = 4e^{2x},
\]
so that \(-12A = 4\) or \(A = -\frac{1}{3}\). Thus \(y_p = -\frac{1}{3}e^{2x}\) and the general solution of (4) is

\[y = y_p + y_h = -\frac{1}{3}e^{2x} + c_1e^{6x} + c_2e^{-x}. \]

Note that differential equations (5) and (6) had exactly the same forcing function \(4e^{2x}\) and the coefficients differed in only one place (-6 vs. 6), yet the form for \(y_p\) was different. This points out an important fact.

In general, the form for \(y_p\) cannot be determined by looking only at the forcing function \(f\). The solution of the homogeneous equation (roots of the characteristic polynomial) must also be considered.

In the examples to follow we shall make frequent use of the fact that
If \( f(x) = f_1(x) + \cdots + f_m(x) \), then there is a particular solution of the form obtained by adding up the forms of the particular solutions for each \( f_i(x) \).

Also, in several of the examples we shall merely determine the form for \( y_p \) and not actually find the constants in the form.

**Special Case 3**

If \( f(x) \) is \( p(x)e^{ax} \), where \( p(x) \) is an \( m \)-th-degree polynomial, then there is a particular solution of the form

\[
x^k[A_0 + A_1x + \cdots + A_m x^m]e^{ax},
\]

where \( k \) is the multiplicity of \( a \) as a root of the characteristic polynomial. Note that Special Case 3 includes Special Case 1 and 2 by taking \( a = 0 \) and \( m = 0 \), respectively.

**Example 3.12.6**

Give the form for \( y_p \) if

\[
y'' - y' = x^3 + x + e^x - 2xe^x
\]

is to be solved by the method of undetermined coefficients.

The characteristic polynomial is \( r^2 - r = r(r - 1) \), which has roots \( r = 0, 1 \) so that \( y_h = c_1 + c_2 e^x \). The forcing term is

\[
f = (x^3 + x) + (1 - 2x)e^x.
\]

The first term is a third-degree polynomial. Since 0 is a root of multiplicity 1 of the characteristic equation (Special Case 1), \( y_p \) must include a term of the form \( x^k(A_0 + A_1 x + A_2 x^2 + A_3 x^3) \) with \( k = 1 \). The second term is of the form \( p(x)e^{ax} \), where \( p(x) = 1 - 2x \) is a first-degree polynomial and \( a = 1 \). Since 1 is a root of the characteristic equation by Special Case 3, \( y_p \) must include a term of the form \( x^k(A_4 + A_5 x)e^x \) with \( k = 1 \). Thus \( y_p \) has the form

\[
y_p = x(A_0 + A_1 x + A_2 x^2 + A_3 x^3) + x(A_4 + A_5 x)e^x.
\]

**Example 3.12.7**

Give the form for \( y_p \) if

\[
y'' - 2y' + y = 7xe^x
\]

is to be solved by the method of undetermined coefficients.

The characteristic polynomial \( r^2 - 2r + 1 = (r - 1)^2 \) has a root 1 of multiplicity 2. Thus \( y_h = c_1 e^x + c_2 xe^x \). The forcing term is of the form \( p(x)e^{ax} \), where \( p(x) = 7x \) is a first-degree polynomial and \( a = 1 \). Since \( a = 1 \) is a root of multiplicity 2, by Special Case 3 with \( k = 2, m = 1 \), the form for \( y_p \) is

\[
x^2(A_0 + A_1 x)e^x.
\]
Special Case 4

If \( f(x) = E_1 \cos \beta x + E_2 \sin \beta x \), where at least one of the constants \( E_1, E_2 \) is nonzero, then \( y_p \) has the form \( x^k(A_0 \cos \beta x + B_0 \sin \beta x) \), where \( k \) is the multiplicity of \( \beta i \) as a root of the characteristic polynomial.

Example 3.12.8

Find the general solution of

\[
y'' + 2y' + 2y = 3e^{-x} + 4 \cos x. \tag{8}
\]

First we solve the associated homogeneous equation. The characteristic equation is \( r^2 + 2r + 2 = 0 \). Its roots are

\[ r = -1 \pm i. \]

Thus \( y_h = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x \). Now we will determine \( y_p \). Note that \( f \) is a sum of two terms, \( 3e^{-x} \) and \( 4 \cos x \). Consider first \( 3e^{-x} \). Since \( -1 \) is not a root of the characteristic equation, Special Case 2 says that \( y_p \) includes a term of the form \( A_1 e^{-x} \). Now consider \( 4 \cos x \). Since \( i \) is not a root of the characteristic equation, Special Case 4 with \( \beta = 1 \) says that \( y_p \) includes terms of the form \( A_2 \cos x + A_3 \sin x \). Thus \( y_p = A_1 e^{-x} + A_2 \cos x + A_3 \sin x \) for some constants \( A_1, A_2, A_3 \). Substituting this expression into (8) gives

\[
[A_1 e^{-x} + A_2 \cos x + A_3 \sin x]'' + 2[A_1 e^{-x} + A_2 \cos x + A_3 \sin x]' + 2[A_1 e^{-x} + A_2 \cos x + A_3 \sin x] = 3e^{-x} + 4 \cos x.
\]

That is,

\[
A_1 e^{-x} - A_2 \cos x - A_3 \sin x - 2A_1 e^{-x} - 2A_2 \sin x + 2A_3 \cos x + 2A_1 e^{-x} + 2A_2 \cos x + 2A_3 \sin x = 3e^{-x} + 4 \cos x.
\]

There are three functions \( e^{-x}, \cos x, \sin x \) that appear in this equation. Equating the coefficients of like terms gives

\[
e^{-x}: \quad A_1 = 3.
\]
\[
\cos x: \quad A_2 + 2A_3 = 4.
\]
\[
\sin x: \quad A_3 - 2A_2 = 0. \tag{9}
\]

This system of three equations in the three unknowns \( A_1, A_2, A_3 \) has the solution

\[ A_1 = 3, \quad A_2 = \frac{3}{2}, \quad A_3 = \frac{3}{2}. \]

Thus \( y_p = 3e^{-x} + \frac{3}{2} \cos x + \frac{3}{2} \sin x \), and

\[ y = y_p + y_h = 3e^{-x} + \frac{3}{2} \cos x + \frac{3}{2} \sin x + c_1 e^{-x} \cos x + c_2 e^{-x} \sin x. \]

This example emphasizes another common source of errors:
Even though only \( \cos \beta x \) appeared in the forcing term \( f \), the form for \( y_p \) may require both \( x^4A \cos \beta x \) and \( x^4B \sin \beta x \).

The system of equations (9) consisted of three equations in three unknowns and had a unique solution. It can be proved, using properties of linear independence and the Wronskian, that

If \( a, \ b, \ c \) are constants and \( f \) is the type of function described in the Method of Undetermined Coefficients, then the method always works (perhaps messily). In particular, if the equations for the undetermined constants are not consistent (don’t have a solution), then an error has been made.

For example, if one arrives at

\[
A_1 \sin x + A_2 e^{-x} = \cos x + 3 \sin x + 5e^{-x},
\]
equating coefficients of all the functions that appear gives

\[
\begin{align*}
\sin x: & \quad A_1 = 3, \\
\cos x: & \quad 0 = 1, \\
e^{-x}: & \quad A_2 = 5,
\end{align*}
\]

which is impossible. Since the method always works for appropriate \( f \), we know that we have made an error. Frequently the error is in finding the form for \( y_p \).

**Example 3.12.9**

Give the form for \( y_p \) if

\[y'' + 4y = \sin 2x\]

is to be solved by the method of undetermined coefficients.

The roots of the characteristic polynomial \( r^2 + 4 \) are \( \pm 2i \) and \( y_k = c_1 \cos 2x + c_2 \sin 2x \). The forcing term \( \sin 2x \) is \( \sin \beta x \), where \( \beta = 2 \). Since \( \beta \) is a root of the characteristic polynomial of multiplicity 1, we have \( k = 1 \) and by Special Case 4, \( y_p \) has the form \( x[A_0 \cos 2x + B_0 \sin 2x] \).

**Solution**

**Special Case 5**

If \( f(x) = p(x) \sin \beta x + q(x) \cos \beta x \), where \( p(x) \) is an \( m \)th-degree polynomial in \( x \) and \( q(x) \) is an \( n \)th-degree polynomial in \( x \), then there is a particular solution of the form

\[x^4[(A_0 + A_1 x + \cdots + A_s x^s) \cos \beta x + (B_0 + B_1 x + \cdots + B_s x^s) \sin \beta x].\]
where \( k \) is the multiplicity of \( \beta i \) as a root of the characteristic polynomial and \( s \) is the largest of \( m, n \).

Special Case 5 includes Special Cases 3 \((m = 0, n = 0)\), and 1 \((\beta = 0, n = 0)\).

**Example 3.12.10**

Give the form for \( y_p \) if

\[
y'' + 4y = x^2 \cos 2x - x \sin 2x + \sin 2x = x^2 \cos 2x + (1 - x) \sin 2x
\]

is to be solved by the method of undetermined coefficients.

The roots of the characteristic polynomial \( r^2 + 4 \) are \( \pm 2i \) and \( y_h = c_1 \cos 2x + c_2 \sin 2x \). The forcing term is of the form

\[ p(x) \cos \beta x + q(x) \sin \beta x, \]

where \( p(x) = x^2 \) is a second-degree polynomial, \( q(x) = 1 - x \) is a first-degree polynomial, and \( \beta = 2 \). Since \( \beta i = 2i \) is a root of the characteristic equation of multiplicity 1 by Special Case 5, with \( k = 1 \), we have

\[
y_p = x[(A_1 + A_2 x + A_3 x^2) \cos 2x + (A_4 + A_5 x + A_6 x^2) \sin 2x].
\]

Our final special case is:

**Special Case 6**

If \( f(x) = E_1 e^{\alpha x} \cos \beta x + E_2 e^{\alpha x} \sin \beta x \), where \( E_1, E_2 \) are constants at least one of which is nonzero, then there is a particular solution of the form

\[ x^k[A_0 e^{\alpha x} \cos \beta x + B_0 e^{\alpha x} \sin \beta x], \]

where \( k \) is the multiplicity of \( \alpha + \beta i \) as a root of the characteristic polynomial.

**Example 3.12.11**

Give the form for \( y_p \) if

\[
y'' + 2y' + 2y = 5e^{-x} \cos x
\]

is to be solved by the method of undetermined coefficients.

The roots of the characteristic polynomial \( r^2 + 2r + 2 \) are \(-1 \pm i\), so that

\[ y_h = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x. \]

The forcing term is of the form \( e^{\alpha x} \cos \beta x \), where \( \alpha = -1, \beta = 1 \). Since \(-1 + i\) is a root of the characteristic equation of multiplicity 1, by Special Case 6, with \( k = 1 \),

\[
y_p = x(A_0 e^{-x} \cos x + B_0 e^{-x} \sin x).\]
Example 3.12.12

Give the form for \( y_p \) if
\[
y'' + 2y' + 2y = e^{-x} \cos 2x + e^{-x} \sin 2x + e^{-x} - 3 \cos x
\]  
(10)

is to be solved by undetermined coefficients.

Equation (10) has the same characteristic polynomial as Example 3.12.11, so the roots are \( -1 \pm i \). The forcing term in (10) is the sum of three groups of terms:

\[
e^{-x} \cos 2x + e^{-x} \sin 2x: \quad \text{Since } -1 + 2i \text{ is not a root, we include } A_0 e^{-x} \cos 2x + B_0 e^{-x} \sin 2x, \quad \text{by Special Case 6.}
\]

\[
e^{-x}: \quad \text{Since } -1 \text{ is not a root, we include } A_2 e^{-x}, \text{ by Special Case 2.}
\]

\[
-3 \cos x: \quad \text{Since } i \text{ is not a root, we include } A_1 \cos x + B_1 \sin x, \text{ by Special Case 4.}
\]

Thus the form for \( y_p \) is
\[
A_0 e^{-x} \cos 2x + B_0 e^{-x} \sin 2x + A_1 \cos x + B_1 \sin x + A_2 e^{-x}. \quad \blacksquare
\]

When initial conditions are present and the forcing function is nonzero, it is important to remember to apply the initial conditions to the general solution \( y_p + y_h \) and not just to \( y_h \).

Example 3.12.13

Solve
\[
y'' - 5y' + 6y = 4e^{2x}, \quad y(0) = 0, \quad y'(0) = 1.
\]  
(11)

First find the general solution of \( y'' - 5y' + 6y = 4e^{2x} \). This was done in Example 3.12.4 and is
\[
y = -4xe^{2x} + c_1 e^{2x} + c_2 e^{3x}.
\]  
(12)

In order to apply the initial conditions, we must compute \( y' \). Differentiate (12) to get
\[
y' = -4e^{2x} - 8xe^{2x} + 2c_1 e^{2x} + 3c_2 e^{3x}.
\]

The initial conditions then give
\[
0 = y(0) = c_1 + c_2, \quad 1 = y'(0) = -4 + 2c_1 + 3c_2.
\]

Solving for \( c_1, c_2 \) yields \( c_1 = -5, c_2 = 5 \), and the solution of (11) is
\[
y = -4xe^{2x} - 5e^{2x} + 5e^{3x}. \quad \blacksquare
\]
3.12.1 The Superposition Principle

The Superposition Principle (Section 3.4) for linear differential equations says that

If $y_{p1}$ is a particular solution of $ay'' + by' + cy = f_1$ and $y_{p2}$ is a particular solution of $ay'' + by' + cy = f_2$, then $y_{p1} + y_{p2}$ is a particular solution of $ay'' + by' + cy = f_1 + f_2$.

Intuitively, the superposition principle means that the response (output) that results from the sum (superposition) of two forcing terms (inputs) is the sum of the response from each forcing term. In applications, knowing whether our device or physical problem acts in this manner is a key factor in deciding whether linear equations can be used to analyze the problem. Fact (7) is one version of the superposition principle.

The superposition principle can also be used as follows: Suppose we wish to solve

$$y'' + 3y' + 2y = x^2e^{-x} + \cos 2x - 3x^2 \sin x.$$  \hspace{1cm} (13)

If the method of undetermined coefficients is applied to (13), the form for $y_p$ will have 11 constants to determine. Alternatively, the superposition principle says if

$y_{p1}$ is a particular solution of $y'' + 3y' + 2y = x^2e^{-x},$

$y_{p2}$ is a particular solution of $y'' + 3y' + 2y = \cos 2x,$

$y_{p3}$ is a particular solution of $y'' + 3y' + 2y = -3x^2 \sin x,$

then $y_p = y_{p1} + y_{p2} + y_{p3}$ would be a particular solution of (13).

Using the superposition principle in this manner will never reduce the amount of calculation. It will, however, often reduce the length of the expressions being worked with and thus reduce human error. If undetermined coefficients are being implemented symbolically on a computer, there is no advantage.

The superposition principle, combined with the theory of Fourier series, enables the method of undetermined coefficients to be used on many additional kinds of forcing functions $f$. The idea is to write $f(x)$ as an (infinite) linear combination of functions $\cos \beta x$, $\sin \beta x$ for different $\beta$, and then use an infinite-series version of the superposition principle. This important idea is somewhat beyond the immediate aims of this book and will not be pursued further.

Undetermined coefficients can also be applied to first-order linear constant-coefficient differential equations if $f$ is in the right form and is often quicker than using integrating factors.

Method of Undetermined Coefficients

In summary, the method of undetermined coefficients can be used on $ay'' + by' + cy = f$ if $a$, $b$, $c$ are constants and $f$ is a linear combination of functions of the form
\[ x^m e^{ax} \cos \beta x, \quad x^m e^{ax} \sin \beta x, \]

where \( m \) is a nonnegative integer and \( a, \beta \) are real numbers. Special cases are:

\[ x^n, \quad x^m e^{ax}, \quad e^{ax} \cos \beta x, \quad e^{ax} \sin \beta x, \quad x^m \cos \beta x, \quad x^m \sin \beta x. \]

The method is as follows.

1. First solve the associated homogeneous equation

\[ ay'' + by' + cy = 0. \]

2. Determine \( y_p \) as a linear combination of functions with unknown coefficients using the following rules.

R1. If \( f \) includes a sum of terms of the form \( p(x)e^{ax} \), where \( p(x) \) is an \( m \)th-degree polynomial, then the form for \( y_p \) should include

\[ x^k[A_0 + A_1 x + \cdots + A_n x^n]e^{ax}, \]

where \( k \) is the multiplicity of \( x \) as a root of the characteristic polynomial \( ar^2 + br + c \).

R2. If \( f \) includes a sum of terms of the form

\[ p(x)e^{ax} \cos \beta x, \quad q(x)e^{ax} \sin \beta x, \]

where \( p(x) \) is an \( m \)th-degree polynomial and \( q(x) \) is an \( n \)th-degree polynomial, then the form for \( y_p \) should include

\[ x^s[B_0 + B_1 x + \cdots + B_n x^n]e^{ax} \cos \beta x \\
+ x^s[B_0 + B_1 x + \cdots + B_n x^n]e^{ax} \sin \beta x, \]

where \( s \) is the larger of \( m \) and \( n \) and \( k \) is the multiplicity of \( x + \beta i \) as a root of the characteristic polynomial \( ar^2 + br + c \).

3. Substitute the expression for \( y_p \) into the differential equation

\[ ay'' + by' + cy = f \]

to determine the unknown coefficients \( A_i, B_i \).

4. The general solution of \( ay'' + by' + cy = f \) is \( y = y_p + y_h \).

5. Apply any initial conditions to \( y_p + y_h \) in order to determine arbitrary constants.

Note that if \( f(x) \) includes terms like \( \ln x, \quad x^{1/3} \sin x, \quad \tan x \), then undetermined coefficients will not generally work.

**Exercises**

In Exercises 1 through 12, state whether undetermined coefficients can be applied to the differential equation. If it cannot, explain why not.

1. \( y'' + y = x \sin x \)
2. \( y'' + 3y = x^{1/3} \sin x \)
3. \( y'' + y = x^2 + x + \ln|x| \)
4. \( y'' + y = e^{x^2} \)
5. \( y'' + y = \frac{\sin x}{\cos x} \)
6. $y'' + y' + y = \cosh x$
7. $y'' + xy' = 3e^{2x}$
8. $y'' + y' = x \sinh 2x$
9. $y'' + y = x^{-1}e^x$
10. $y'' + yy' = e^{2x}$
11. $y'' + 3y = e^{-2x} \cos 3x + \sinh 3x$
12. $y'' + 3y' + 4y = \sin 3x$

In Exercises 13 through 36, solve the differential equation using the method of undetermined coefficients. If no initial conditions are given, give the general solution.

13. $y'' - 3y' + 2y = 2e^x$
14. $y'' - 3y' + 2y = 2e^{-x}$
15. $y'' + 2y' + 5y = 3 \sin x$
16. $y'' + 4y' + 8y = x^2 + 1; \quad y(0) = 0, \quad y'(0) = 0$
17. $y'' + y = \cos 2x; \quad y(0) = 0, \quad y'(0) = 1$
18. $y'' - 4y' + 3y = xe^x$
19. $y'' - 4y' + 3y = xe^x$
20. $y'' + y = \sin x$
21. $y'' - 2y + 5y = e^x \cos 3x$
22. $y'' + 3y' + y = 3e^x; \quad y(0) = 0, \quad y'(0) = 2$
23. $y'' + 4y' + 12x^2 + e^x; \quad y(0) = 1, \quad y'(0) = 1$
24. $y'' + y' + y = \cos 2x$
25. $y'' + 2y' + y = 3e^{-x}$
26. $y'' + 2y' + y = 3xe^{-x} + 2e^{-x}$

27. $y'' - 4y' + 4y = xe^x - e^x + 2e^{3x}$
28. $y'' - 5y' + 4y = 17 \sin x + 3e^{2x}$
29. $y'' + 5y' + 4y = 8x^2 + 3 + 2 \cos 2x$
30. $y'' + y = 2e^{-x}$
31. $y'' + 3y = x^2 + 1$
32. $y'' - y = \sin x$
33. $y'' + 4y = \sin 2x$
34. $2y'' + 4y = x$
35. $3y'' - 2y = xe^x$
36. $y'' - 3y = e^x \sin x$

In Exercises 37 through 49, give the form for $y_p$ if the method of undetermined coefficients were to be used. You need not actually compute $y_p$.

37. $y'' - 2y' + 5y = 3e^x \sin 2x$
38. $y'' + 2y' - 3y = x^2 e^x - e^x + e^{-2x} + e^{-3x}$
39. $y'' - 6y + 9y = x^2 e^{3x}$
40. $y'' + 9y = x^2 \sin 3x + \cos 2x$
41. $y'' + 9y = xe^{-x} \sin 2x$
42. $y'' + 2y' + 2y = 3x^2 e^{-x} \cos 2x + xe^{-x} \sin 2x$
43. $y'' + 2y' + 2y = e^{-x} \cos x + e^x \sin x$
44. $y'' + 3y' - 10y = x^4 \cos x$
45. $y'' + 3y' - 10y = x^2 e^{2x} + e^{4x}$
46. $y'' + y = e^{-x} - e^x + e^x \cos x$
47. $y'' + 16y = x \cos 4x + e^{-x} \sin 4x + 3e^{-4x}$
48. $y'' + 3y' + 2y = x^2 e^{-x} \cos x$
49. $y'' - 2y' + 2y = x^2 e^{-x} \sin x + e^x \cos x$

### 3.13

**Undetermined Coefficients (nth-Order)**

#### 3.13.1

**The Procedure**

The method of undetermined coefficients described in Section 3.12 works with only slight modification on nth-order linear differential equations with real constant coefficients:
\[ a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_1 y'(x) + a_0 y(x) = f(x). \]  
(1)

The forcing or input function \( f(x) \) still needs to be a linear combination of functions of the form \( x^m e^{\alpha x} \cos \beta x \), \( x^m e^{\alpha x} \sin \beta x \) for integers \( m \geq 0 \) and real numbers \( \alpha, \beta \). There is only one change in the method of undetermined coefficients from the second-order case.

The characteristic polynomial is now \( n \)th-degree, so that its roots may have multiplicity greater than two.

We still have

R1'. If \( f(x) \) includes a sum of terms of the form \( p(x)e^{\alpha x} \), where \( p(x) \) is an \( m \)th-degree polynomial, then the form for \( y_p \) should include

\[ x^k [A_0 + A_1 x + \ldots + A_m x^m] e^{\alpha x} \]

where \( k \) is the multiplicity of \( \alpha \) as a root of the characteristic polynomial.

R2'. If \( f(x) \) includes a sum of terms of the form

\[ p(x)e^{\alpha x} \cos \beta x + q(x)e^{\alpha x} \sin \beta x, \]

where \( p(x) \) is an \( m \)th-degree polynomial and \( q(x) \) is an \( n \)th-degree polynomial, then the form for \( y_p \) should include

\[ x^k [A_0 + A_1 x + \ldots + A_s x^s] e^{\alpha x} \cos \beta x + x^k [B_0 + B_1 x + \ldots + B_n x^n] e^{\alpha x} \sin \beta x, \]

where \( s \) is the larger of \( m \) and \( n \) and \( k \) is the multiplicity of \( \alpha + \beta i \) as a root of the characteristic polynomial.

**Example 3.13.1**

Find the general solution of

\[ y''' - 6y'' + 12y' - 8y = xe^{2x}. \]  
(2)

**Solution**

First we need to solve the associated homogeneous equation

\[ y''' - 6y'' + 12y' - 8y = 0. \]

The characteristic polynomial is \( r^3 - 6r^2 + 12r - 8 = (r - 2)^3 \), so that 2 is a root of multiplicity 3. A fundamental set of solutions for the associated homogeneous equation is

\[ \{e^{2x}, xe^{2x}, x^2 e^{2x}\}. \]

The forcing term is \( f(x) = xe^{2x} \). By Rule R1' with \( \alpha = 2, m = 1 \), \( y_p \) must include \( x^k [A_0 + A_1 x] e^{2x} \), where \( k \) is the multiplicity of 2 as a root of the characteristic equation. Thus \( k = 3 \) and \( y_p \) is in the form

\[ y_p = x^3 [A_0 + A_1 x] e^{2x} = A_0 x^3 e^{2x} + A_1 x^4 e^{2x}. \]

To find \( A_0, A_1 \), substitute the form for \( y_p \) into the original differential equation (2):
\[(A_0 x^3 e^{2x} + A_1 x^4 e^{2x})''' - 6(A_0 x^3 e^{2x} + A_1 x^4 e^{2x})'' + 12(A_0 x^3 e^{2x} + A_1 x^4 e^{2x})' - 8(A_0 x^3 e^{2x} + A_1 x^4 e^{2x}) = xe^{2x}.\]

After differentiating and combining like terms, we get
\[6A_0 e^{2x} + (30A_0 + 24A_1)xe^{2x} = xe^{2x}\]

Thus equating coefficients of like terms:
\[e^{2x}: \quad 6A_0 = 0,\]
\[xe^{2x}: \quad 30A_0 + 24A_1 = 1\]

and \(A_0 = 0, A_1 = \frac{1}{24}\). Thus \(y_p = \frac{1}{24}x^4 e^{2x}\) and the general solution of (2) is
\[y = \frac{1}{24}x^4 e^{2x} + c_1 e^{2x} + c_2 xe^{2x} + c_3 x^2 e^{2x}.
\]

**Example 3.13.2**

Give the form for \(y_p\) if
\[y''' + 8y'' + 16y = x \sin x + x^2 \cos 2x\]

(3)

is to be solved by the method of undetermined coefficients.

First we solve the associated homogeneous equation
\[y''' + 8y'' + 16y = 0.\]

The characteristic equation is \(r^3 + 8r^2 + 16 = (r^2 + 4)^2 = 0\) and has repeated complex roots \(\pm 2i, \pm 2i\). A fundamental set of solutions for the associated homogeneous equation is thus
\[
\{\sin 2x, \cos 2x, x \sin 2x, x \cos 2x\}.
\]

Now we find the form for \(y_p\). Consider the forcing function
\[f = x \sin x + x^2 \cos 2x.\]

By R2', the \(x \sin x\) term implies that \(y_p\) includes
\[x^k [(A_0 + A_1 x) \sin x + (B_0 + B_1 x) \cos x]\]

(4)

with \(k_1 = 0\) since \(i\) is not a root of the characteristic polynomial. The \(x^2 \cos 2x\) term implies that \(y_p\) includes
\[x^k [(A_2 + A_3 x + A_4 x^2) \sin 2x + (B_2 + B_3 x + B_4 x^2) \cos 2x]\]

(5)

with \(k_2 = 2\), since \(2i\) has multiplicity two as a root of the characteristic equation. In actually using (4) and (5), one may add (4) and (5) to get the form for \(y_p\), substitute into the original differential equation (3), and solve for \(A_0, \ldots, A_4, B_0, \ldots, B_4\). Alternatively, one could use (4) to find a particular solution of
\[y''' + 8y'' + 16y = x \sin x,\]

and then use (5) to find a particular solution of
\[ y''' + 8y'' + 16y = x^2 \cos 2x. \]

Adding these two particular solutions gives, by the superposition principle, a solution of \( y''' + 8y'' + 16y = x \sin x + x^2 \cos 2x \) as desired. \( \blacksquare \)

### Exercises

In Exercises 1 through 10, solve the differential equation by the method of undetermined coefficients. If no initial conditions are given, give the general solution.

1. \( y''' - 3y'' + 3y' - y = e^{2x} \)
2. \( y''' - 3y'' + 3y' - y = e^x \)
3. \( y''' - y' = \sin x \)
4. \( y''' - 25y'' + 144y = x^2 - 1 \)
5. \( y''' + y' = 3 + 2 \cos x \quad y(0) = y'(0) = y''(0) = 0 \)
6. \( y''' - y = 2e^{3x} - e^x \)
7. \( y''' - 16y = 5xe^x \)
8. \( y''' + 4y'' + 4y = \cos 2x \)
9. \( y''' - 5y'' + 4y = e^{2x} - e^{3x} \)
10. \( y''' + y'' - 6y'' = 72x + 24, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -6, \quad y'''(0) = -57 \)

In Exercises 11 through 18, give the form for \( y_p \) that you would use to find a particular solution by the method of undetermined coefficients. Do not actually solve for \( y_p \).

11. \( y''' - 3y'' + 3y' - y = x^2 e^x - 3e^x \)
12. \( y''' + 2y' + y = x \sin x \)
13. \( y''' - 4y'' + 6y' - 4y + y = x^2 e^x + x^2 e^{-x} \)
14. \( y''' + 5y'' + 4y = \sin x + \cos 2x + \sin 3x \)
15. \( y''' + 2y' + 2y = 3e^{-x} \cos x \)
16. \( y''' + 2y' + 2y' = x^2 e^{-x} \cos x - xe^{-x} \sin x \)
17. \( y''' + 4y'' + 8y' + 8y + 4y = 7e^{-x} \cos x \)
18. \( y''' - 2y'' + y = xe^x + x^2 e^{-x} + e^{2x} \)

### 3.13.2

Undetermined Coefficients Using Annihilators (Requires Section 3.6)

As noted in Sections 3.12 and 3.13, the method of undetermined coefficients always works if we have a linear constant-coefficient differential equation and the forcing function \( f \) is of the right kind. This section will provide one explanation of why that is true. If can also be used as an alternative approach for carrying out the method of undetermined coefficients.

Suppose that we have the linear differential equation with constant coefficients

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f \]

or

\[ L_1 y = f, \]
Section 3.13
1. $y = e^{2x} + c_1 e^x + c_2 xe^x + c_3 x^2 e^x$
2. $y = \frac{1}{2} \cos x + c_1 + c_2 e^x + c_3 e^{-x}$
3. $y_p = A_1 x^2 + B_1 x + C_1 x + D_1 x^2 + E_1 x + F_1$
4. $y_p = -x \cos x + 2x - 3 \sin x$
5. $y_p = A x + B x \cos x + C x \sin x$
6. $y_p = -x \cos x + 3x - 2 \sin x$
7. $y_p = \frac{4}{45} e^{x} - \frac{1}{3} x e^x + c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$
8. $y_p = -\frac{1}{40} e^{3x} + \frac{1}{12} x e^{2x} + c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$
9. $y_p = A_0 x^3 + B_0 x^2 + C_0 x + D_0 x + E_0$
10. $y_p = A_1 x + B_1 x \cos x + C_1 x \sin x$ (Note: In the solutions for Exercises 19 through 25, subscript on constants may vary.)
11. $(D^2 + 1) y_p = c_1 x^2 + c_2 x^2 + c_3 x$ cos x
12. $(D^2 + 1)^2 y_p = c_1 x^2 + c_2 x^2 + c_3 x$ cos x
13. $(D^2 + 1)^3 y_p = c_1 x^2 + c_2 x^2 + c_3 x$ cos x
14. $(D^2 + 1)^4 y_p = c_1 x + c_2 x + c_3 x^2 + c_4 x^4$

Section 3.14
1. $y = \frac{1}{3} x^2 + c_1 e^x + c_2 e^{-x}$, yes
2. $y = \frac{1}{3} x^2 + c_1 e^x + c_2 e^{-x}$, yes
3. $y = (\sin x) \ln |\sin x| - x \cos x + c_1 \sin x + c_2 \cos x$, no
4. $y = -x \cos x + (c_1 + c_2) \cos x + (c_3 + c_4) \sin x + c_5$ cos x
5. $y = -x \cos x - (c_1 + c_2) \cos x + c_3$ cos x
6. $y = -x \cos x + (c_1 + c_2) \cos x + c_3$ cos x

11. $y = \frac{1}{4} x e^{-x} - \frac{x}{4} e^{-x} + c_1 e^{-x} + c_2 x e^{-x}$
12. $y = \frac{1}{4} x e^{-x} + c_1 e^{-x} + c_2 x e^{-x}$, no
13. $y = \left(\frac{1}{25} e^{-x}\right) e^{3x} - \frac{1}{5} x e^{-x} + c_1 e^{3x} + c_2 e^{-x} = \frac{1}{5} x e^{3x} + c_1 e^{3x} + c_2 e^{-x}$, yes
14. $y = \frac{1}{2} x^3 + c_1 x + c_2 x^2$
15. $y = \frac{1}{2} x^3 + c_1 x + c_2 x^2$
16. $y = -x^{-1} - x^{-1} \ln x + c_1 + c_2 x^{-1} = -x^{-1} \ln x + c_1 + c_2 x^{-1}$
17. $y = \frac{1}{s + 1} \left[ e^{2(t-s)} - e^{2(t-s)} \right] \int_0^s \frac{1}{s + 1} ds$
18. $K f = \int_0^s f(s) ds$
19. $g = K(L f) = L(g) = g = f + y_n$, and $g(0) = 0$, $g'(0) = 0$. Solve for constants in $y_n$.