FURTHER REMARKS, PARTICULAR FLOWS & LIMITING CASES

The stress tensor: Recall that the surface force on a surface element $dS$ of unit normal $n$ is given as $(dS)Tn$ in plane. $T$ is the stress tensor. The terminology "tensor" is used to indicate that this is an object (in this case a 4x4, if x, y, t, & u) which is independent of the coordinate system in which we happen to express it. Now, if we fix an orthogonal (for convenience) coordinate system we can realize $T$ as a matrix in this basis:

\[ T = \begin{pmatrix}
  T_{11} & T_{12} & T_{13} \\
  T_{21} & T_{22} & T_{23} \\
  T_{31} & T_{32} & T_{33}
\end{pmatrix} \]

Let's interpret this by looking at the stress experiences by unit areas in the $xy$-, $xz$-, and $yz$-planes (when we fix an orthogonal $xyz$-coord. system):
Here: \( \mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) s.d., \( T_n = \begin{pmatrix} T_{n1} \\ T_{n2} \end{pmatrix} \)

So, for example:

\( T_{11} = 1\)-component of force felt on a unit surface element whose normal points in the positive 1-direction.

\( T_{21} = 2\)-comp. of force felt on a unit surface element whose normal points in the positive 1-direction.

etc.

In general:

\( T_{ij} = i\)-comp. of force experienced by a unit area whose normal is in the pos. \( j \)-dir.

Thus: off-diagonal elements of \( T \) represent tangential stresses.
Perfect fluids: real fluids can exert tangential stresses, so that $T$ is typically not diagonal. In some situations, however, tangential stresses may be small. The limiting case is called a perfect fluid, in which case $T$ is diagonal:

$$T_{n} = -p n$$

where the scalar $p = p(x, t)$ is called pressure.

In this case, the momentum eq. takes the form:

$$\frac{\partial u}{\partial t} = -\nabla p + \mathbf{f}.$$

We will get back to this again when we consider the eq.'s for gas-dynamics.

Entropy & internal energy:

We postulate that the total (specific) energy $E$ is given as the sum of internal energy $u$ (a thermodynamical quantity)
and kinetic energy $\frac{1}{2} |u|^2$ (a kinematic quantity).

Now, by taking the inner product of the momentum $u \cdot w$ and subtract the result from the energy $e$, we obtain the following equation for internal energy:

$$g(\rho) + \text{div}(\rho \text{w}) + p \text{div} u = K \Delta \theta + \lambda (\text{div} u)^2 + \frac{1}{2} \left( \partial_i u \partial_j u \right) \eta_{ij} \,.$$

Using the continuity equation, we can write this as:

$$g \frac{de}{dt} = -p \text{div} u + K \Delta \theta + \lambda (\text{div} u)^2 + \frac{1}{2} \left( \partial_i u \partial_j u \right) \eta_{ij} \,.$$

**Note:** The last term sums over both $i$ and $j$!

Now recall that we assume the existence of a thermodynamical quantity $S$, called entropy, which satisfies ($\nu = \text{spec. rel.} = Vg$):

$$\frac{dS}{dc} = 0 \quad \text{and} \quad \frac{dS}{dV} = \frac{dS}{dc} \frac{dc}{dV} = -\frac{1}{g} \,.$$

These are consequences of the $1^{st}$ law of thermodynamics:

$$dS = dc + pdv.$$
We thus get
\[ S = S(v, e) = S(\frac{\dot{v}}{e}, e) \]
\[
\frac{dS}{dt} = -g \left( \frac{25}{dV} \frac{de}{dt} + \frac{25}{de} \frac{dv}{dt} \right)
\]
\[
= -\frac{1}{\theta} \left( -g \delta \nabla \cdot u \right) + \frac{1}{\theta} \left\{ \begin{aligned}
- \delta \nabla \cdot u + \lambda \Delta \theta + \frac{1}{2} \left( \nabla \cdot u \right)^2 \\
+ \frac{e}{\theta} \left( \delta \cdot u + \delta \cdot u \right)^2
\end{aligned} \right. \]
\]
(30.1)
such that (by the cont. eq.):
\[
(\frac{dS}{dt} + \nabla \cdot (\delta \nabla u)) = \frac{\delta S}{dt} = -\frac{1}{\theta} \left\{ \begin{aligned}
\lambda \nabla \cdot u \cdot \nabla \cdot u + \frac{1}{2} \left( \delta \cdot u + \delta \cdot u \right)^2
\end{aligned} \right. \]
\[
(30.2)
\]
\[
\kappa \delta \nabla \cdot (\kappa \delta \nabla \theta) + \frac{\kappa}{\theta} \ln \theta
\]

Suppose the fluid is contained in a fixed \( \Omega \), and that it satisfies the following boundary cond's:

\[
\begin{cases}
\text{no-slip: } u = 0 \text{ on } \partial \Omega \\
\text{thermally insulated: } \nabla \cdot n = \frac{\delta \theta}{\delta n} \equiv 0 \text{ on } \partial \Omega,
\end{cases}
\]

Then integrating over \( \Omega \) yields (by \( \frac{\delta \cdot u \cdot \delta \cdot u}{\delta \nabla \cdot u} \))
\[
\frac{d}{dt} \left( \int_{\Omega} (\delta S) \delta \cdot u \, dx \right) = \int_{\Omega} \frac{1}{\theta} \left\{ \lambda (\nabla \cdot u)^2 + \frac{1}{2} \left( \delta \cdot u + \delta \cdot u \right)^2 \right\} \, dx
\]
(30.3)
An analysis shows that the RHS of (30.3) is $\geq 0$ for all possible flows provided: $\kappa \geq 0$, $\mu \geq 0$ \& $\lambda + \frac{\gamma}{\gamma - 1} \mu > 0$.

Under these conditions it follows that the total entropy is increasing in time (which is a version of the 2nd law of thermodynamics).

Basic a priori information: We pause to collect the basic integral estimates that are direct consequences of the conservation laws. Let $\Omega$ be fixed & assume no slip & thermal insulating BC's on $\partial \Omega$. Then:

$$\int_\Omega \rho \frac{\partial u}{\partial t} \, dx = C \quad \text{(mass)}$$

$$\int_\Omega (p \varepsilon) \, dx \leq C(\varepsilon) \quad \text{(energy)}$$

$$\int_\Omega \frac{\partial}{\partial t} \left[ \frac{\kappa}{\gamma - 1} \frac{\partial u}{\partial x} \right] \, dx + \int_\Omega \frac{\gamma}{2} \left[ \frac{\partial \rho}{\partial x} \right]^2 + \frac{1}{2} \left[ \frac{\partial \rho}{\partial y} \right]^2 + \frac{1}{2} \left( \frac{\partial \rho}{\partial z} \right)^2 \, dx + C_4 \quad \text{(entropy)}$$
homogeneous fluids where \( \gamma \) is const. throughout.

The mom. eq. now takes the form: (assuming const. viscosity)

\[
\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f} - \nabla p + \mu \Delta \mathbf{u},
\]  

(33.1)

while the energy eq. takes the form: (internal energy & const. transport coeff)

\[
\frac{\partial \varepsilon}{\partial t} = \kappa \Delta \theta + \frac{1}{2} (\partial u \partial + \partial v \partial)^2
\]  

(33.2)

For a discussion of incompressible flow, see Chorin & Marsden.

Inviscid flow: If we disregard the effects of molecular transport (of momentum & energy), i.e. we set \( \mu = \lambda = \kappa = 0 \), then we obtain the so-called compressible Euler eqs:

\[
\begin{cases}
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \text{div} (\mathbf{u} \mathbf{u}) &= 0 \\
(\mathbf{u}^T \mathbf{u})_t + \text{div} (\mathbf{u} \mathbf{u} \mathbf{u}) + \nabla p &= \mathbf{f} \\
(\mathbf{S}^T \mathbf{S})_t + \text{div} (\mathbf{S}^T \mathbf{S} \mathbf{u}) &= \mathbf{f} \cdot \mathbf{u}
\end{aligned}
\end{cases}
\]  

(33.3)
This is a system which (presumably) describes well the flow of gases (small visc' & heat cond.). A typical choice for $p$ is the ideal gas law:

$$p = R \rho \Theta.$$

The entropy eq. in this case is

$$\frac{\partial S}{\partial t} + u \cdot \nabla S = 0 \quad \text{or simply:} \quad \frac{dS}{dt} = 0 \quad (1)$$

We will get back to the validity of this eq.

**One-dimensional flow:**

In 1-D the full NS eqs. take the form:

$$\left\{ \begin{array}{l}
\frac{\partial t}{\partial t} + (gu)_x = 0 \quad \frac{\partial g}{\partial t} = -gu_x \\
\frac{\partial g}{\partial t} = -x - (2 \rho) \nabla_{xx} + \frac{\partial g}{\partial x} \\
\frac{\partial \Theta}{\partial t} = - (u \Theta)_x + (x + 2 \rho) \nabla_{xx} + \frac{\partial g}{\partial x}
\end{array} \right. \quad 1 \text{- D, compressible NS w/ const.}

\left[ \text{Internal energy eq.} \right. \frac{\partial E}{\partial t} = -pux + K\Theta_{xx} + (x+2\rho)ux^2$
The 1-D Euler eqs. take the form:

\[
\begin{align*}
\frac{\partial}{\partial t} + \frac{\partial}{\partial x} (pu) &= 0 \\
\frac{\partial}{\partial t} (\rho u) &= -\frac{\partial p}{\partial x} \\
\frac{\partial e}{\partial t} &= -(pu)_x
\end{align*}
\]

(Internal energy \(e\) : \(\frac{\partial e}{\partial t} = -pu_x\))

Lagrangian coordinates for 1-D flow:

Consider the system for 1-D compressible NS w/ cond. transport coeff. Assuming that \(g > 0\) everywhere we can define a new coordinate \(h\) by fixing some reference point, at \(x = 0\) & defining:

\[
h = h(x,t) := \int_{x}^{\infty} g(s,t) \, ds.
\]

I.e. \(h(x,t)\) gives the (signed) mass of fluid between \(a\) & \(x\).
Note: without external forces, the CFD's came consist of only on the initial data. Under reasonable cond's or f. the CFD's will be bounded for's.

An existence theory for the compressible NS eqs. w/ "general" data is currently lacking. A major obstruction is the fact that the above est's are all that seem to hold....

Incompressible flow: The flow is called incompressible provided \( \text{div} \mathbf{u} = 0 \) throughout the fluid. In this case the contrl. takes the form:

\[
\frac{df}{dt} = S_t + \text{div} \mathbf{f} = 0, \tag{32.1}
\]

i.e. the density of a fluid "packet"/particle is constant as it flows along its trajectories. A particular case occurs for
We will assume that \( x \) varies over some interval \( S^2 = (a,b) \), with \( u(x,t) = 0 \) \((a \text{ and/or } b \text{ could be } \infty)\).

We now want to express the eq's in terms of the coordinates \((h,t)\) instead of \((x,t)\). To avoid confusion we define for's \( R, h, E \) by

\[
\begin{align*}
q(x,t) &= R(h(x,t),t), \quad v = \frac{1}{R} \\
\n\end{align*}
\]

\[
\begin{align*}
u(x,t) &= U(h(x,t),t) \\
\end{align*}
\]

\[
\begin{align*}
E(x,t) &= E(h(x,t),t) \\
\end{align*}
\]

\[
\begin{align*}
y(x,0) = 0 \\
\end{align*}
\]

By (35.1): \( h_x = \frac{1}{\varepsilon} \int_a^b \varepsilon \, dx = -\int_a^b (\varepsilon u_x) \, dx = -\varepsilon u \)

\[
\begin{align*}
h_x = q \\
\end{align*}
\]

Thus: if \( q(x,t) = Q(h(x,t),t) \) then

\[
\begin{align*}
\frac{d q}{dt} &= q_t + u \partial_x q \\
&= q_t + u h_x Q_h + u h_x Q_h = q_t + (-\varepsilon u) Q_h + u \varepsilon Q_h \\
&= q_t \\
\end{align*}
\]

Also: \( q = h_x Q_h = e Q_h = R Q_h \).
This really says:

\[ q_{,x}(x, t) = R(h(x, t), t) Q_{h}(h(x, t), t), \]

whence:

\[
\frac{q}{f_{xx}} = R_{x} h_{x} Q_{h} + R Q_{hh} h_{x} \]
\[
= R(R_{x} Q_{h} + R Q_{hh}) = R(R Q_{h})_{,h} = R \left( \frac{Q_{h}}{h} \right). \]

From the full 1-D N5 eqs, we thus get:

Mass:

\[
R_{t} = - R \cdot RU_{h} \quad \rightarrow \quad \frac{R_{t}}{R_{h}} = U_{h} \quad \rightarrow \quad \frac{v}{U} = U_{h}.
\]

Momentum:

\[
RU_{t} = RF - RP_{h} + v R(RU_{h})_{,h}, \quad \text{where:} \quad \int f(x, t) = F(h(x, t), t) \]
\[
p(x, t) = P(h(x, t), t) \]
\[
\lambda = \lambda + 2 \mu \]
\[
U_{t} = F - P_{h} + v \left( \frac{RU_{h}}{V_{h}} \right)
\]

Energy:

\[
RE_{t} = R \cdot R(T_{h})_{,h} - R( RU_{h} ) + v R( U_{h} RU_{h} )_{,h} + REU
\]
\[
\downarrow
\]
\[
E_{t} = \Theta \left( \frac{T_{h}}{V_{h}} \right) - ( RU_{h} ) + v \left( \frac{U_{h} U_{h}}{V_{h}} \right) + FU, \quad \text{where} \quad \Theta(x, t) = T(h(x, t))
\]
It is customary (but confusing) to use the same symbols for both independent var's \((t,x)\) & dependent var's \((R,U,E)\) etc., as in the Eulerian formulation.

Thus the full, 1-D compressible Nav-Stokes eq's in Lagrangian coord's are:

\[
\begin{align*}
u_t - u_x &= 0 \\
\frac{u_t + P_x}{\nu} &= v(u_x) \\
E_t + (Up)_x &= \left( \frac{\nu T_x + U^2 u_x}{\nu} \right)_x
\end{align*}
\]

Where:

- \( \nu = \frac{1}{\gamma} = \text{spec. vol.} \)
- \( u = \text{velocity} \)
- \( p = \text{pressure} \)
- \( E = \epsilon + \frac{u^2}{2} = \text{total energy} \)
- \( T = \text{temperature} \)
- \( \nu = \gamma + 2p \) (assumed const.)
For completeness let's record the inviscid eqs. in Lagr. coord's:

\[ \frac{u_t}{c} - u_x = 0 \]  \hspace{1cm} \text{1-D compressible Euler eqs.'}

\[ u_t + p_x = 0 \]  \hspace{1cm} \text{for gas-dynamics}

\[ E_t + (up)_x = 0 \]  \hspace{1cm} \text{(in Lagrangian coord's)}.

Note: the above derivation requires \( \varphi > 0 \) such that the correspondence between the Eulerian space coordinate \( x \) & the Lagrangian "mass" coordinate \( n \) is one-to-one.

**Barotropic models:** we observe that if \( \varphi \) is one constant, the only way in which the energy eq. is coupled to the mass & momentum eqs' is through the pressure \( p \) in the momentum eq. In particular, provided \( p \) only depends on \( \varphi \),
then we get that the mass & momentum eqs. form a closed system for \( \rho \times u \):

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0 \\
(\rho u)_t + \text{div}(\rho uu) + \text{grad} p = (\lambda + \mu) \nabla^2 (\text{div} u) + \mu \Delta u + ff
\end{cases}
\]  

Such models are called barotropic. Examples are provided by isentropic or isothermal flow.

The inviscid version of (40.1) is

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0 \\
(\rho u)_t + \text{div}(\rho uu) + \text{grad} p = ff
\end{cases}
\]  

which is often called the \( \rho \)-system.

In 1-D this is:

\[
\begin{cases}
\phi_t + (\phi u)_x = 0 \\
(\rho u)_t + (\rho u^2 + p(b))_x = 0
\end{cases}
\]  

\( \text{In Lagrangian coords:} \)

\[
\begin{cases}
\phi_t - \phi u_x = 0 \\
\phi_t + \phi \phi_x = 0 \quad (\phi = p(V))
\end{cases}
\]