Existence of solutions to 1-D compressible Navier-Stokes equations. (After Hoff.)

We will outline an existence result for 1-D compressible NS-\(e\)qs. For simplicity we consider barotropic flow, such that the \(e\)qs. are

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2 + p) &= \varepsilon \Delta u
\end{align*}
\]

(Here \(\varepsilon = \lambda^2 + 2\mu > 0\); we assume zero external forces, and constant transport coefficients.)

We will consider the Cauchy problem, i.e. the initial value problem for (1.1) - (1.2) with initial data \((\rho, u)_{1} = (\rho_0, u_0)\) given on all of \(\mathbb{R}\).
The final result will be the following theorem:

**Thm:** Assume \( T \) is s.t. \( P(s), P'(s) \geq 0 \) for \( s \geq 0 \), and that \( \exists \) constants \( \bar{g}, C \) s.t. the data (p.m.) satisfy:

\[
\frac{1}{C} \leq g(x) \leq C \quad \forall x \in \mathbb{R}, \tag{2.1}
\]

\[
g_0 - \bar{g}, u_0 \in L^2(\mathbb{R}). \tag{2.2}
\]

Then there exists a global weak solution of the system (1.1) - (1.2) (with certain properties).

There are various ways to go about proving this:

1st method: approximate \( g_0, u_0 \) by smooth func's \( g^\varepsilon, u^\varepsilon \) (e.g. by convolution).

- appeal to some "soft" result (proved e.g. by a contraction mapping argument)
- which guarantees short-time existence for smooth data (this is standard, and we will not consider the details of this step).

Let this local soln. be denoted \( (g^\varepsilon, u^\varepsilon) \).
- establish a priori estimates for $(g^5, u^5)$, i.e. certain bounds on $(g^5, u^5)$ which are independent of $\delta$.

These bounds are then used to do two things:

1. Show that $(g^5, u^5)$ actually exists $\forall x < \forall t \geq 0$.
2. The limit $(g, u) = \lim_{\delta \to 0} (g^5, u^5)$ exists, and $(g, u)$ is a weak solution.

2nd Method: Discretize in $x$ & solve ODEs in $t$

- Derive the same type of a priori estimates as above, but now for the discretized approx.
- Pass to the limit as $\Delta x \to 0$.

The second method has the advantage that local existence of the approx. soln. is immediate from standard ODE theory, i.e. there is no need to appeal to a "softer" theory as in the first method (although the local existence theory is not terribly "deep" it still takes a lot of work to write out in detail). On the other
Hand, the second method is notationally messy. For this reason we will adopt Method 1, but we'll skip the proof of local existence of a smooth soln. for smooth data. However, we will go through the all-important a priori bounds in some detail.

4. a priori estimates:

In the following we assume all quantities are regular for all computations to make sense (this is true for the smooth soln.'s $u^e, \dot{u}^e$).

A. This estimate is immediate: let $x_i, \dot{x}_i$ be particle paths,

\[ x_i(t) = U(x_i(0), t) \quad (i = 1, 2). \]

Then:

\[ \int x_i(t) \, dx_i = \text{const. (indep. of time)} \quad (\text{3.1}) \]
\[ \frac{d}{dt} \int_{x_i(t)}^{x_f(t)} \left( \rho(x,t) \frac{\partial u}{\partial t} \right) dx + \int_{x_i(t)}^{x_f(t)} (\rho(x,t) \frac{\partial u}{\partial x})(\frac{\partial u}{\partial x}) dx = \int_{x_i(t)}^{x_f(t)} (\rho u \frac{\partial u}{\partial x})(\frac{\partial u}{\partial x}) dx \]

\[ = - \left[ \rho u \frac{\partial u}{\partial x} \right]_{x_i(t)}^{x_f(t)} = 0. \]

\[ \text{q.e.d.} \]

B. Energy estimate:

First simplify momentum eq. by using mass eq. \((\rho = \rho_0)\)

\[ u \frac{\partial}{\partial t} \left( \rho (u + u u_x) \right) + \rho u \frac{\partial u}{\partial x} = \epsilon u_{xx} \]

\[ \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + \rho \frac{\partial u}{\partial x} = \epsilon u_{xx} \]

\[ \downarrow \]

\[ \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + \rho \frac{\partial u}{\partial x} = \epsilon \left( u u_x \right)_x - \epsilon u_x^2 \]

\[ \downarrow \text{Add mass eq.:} \left[ \rho_t + (\rho u)_x \right], \frac{\partial u}{\partial t} = 0 \]

\[ \left( \rho \frac{u_x^2}{2} + \left( u - \frac{\rho u^2}{2} \right)_x \right) + \rho \frac{\partial u}{\partial x} = \epsilon \left( u u_x \right)_x - \epsilon u_x^2 \]

\[ \downarrow \int dx, \left\{ \begin{array}{c} \frac{u_x}{u} \rightarrow 0 \text{ at } \pm \infty \end{array} \right. \]

\[ dt \left( \int_{-\infty}^{\infty} \frac{u_x^2}{2} dx \right) + \int_{-\infty}^{\infty} \rho \frac{\partial u}{\partial x} dx + \epsilon \int_{-\infty}^{\infty} u_x^2 dx = 0 \]
Next; recall \( \bar{g} = \) reference density (constant), & define:

\[
G(g) = \frac{1}{g} \int \frac{P(s) - \bar{P}}{s^2} \, ds
\]

(The way seem artificial but it actually has a natural interpretation as the potential energy in the gas.)

Then:

\[
G'(g) = \frac{G}{s} + \frac{p - \bar{p}}{s}
\]

\[
\left\{ \begin{array}{l}
p = P(g) \frac{g}{s} \\
\bar{p} = \bar{P}(g)\end{array} \right.
\]

s.t. multiplying the mass eq. by \( G'(g) \) yields:

\[
G'(g) \cdot \frac{g}{s} \dot{\mathbf{u}} + \mathbf{g} \mathbf{u} + \mathbf{g} \mathbf{u}_x = 0
\]

\[
\Rightarrow \quad G(g) \frac{d}{dt} - G(g) \mathbf{u} + (G + p - \bar{p}) \mathbf{u}_x = 0
\]

\[
\Rightarrow \quad G(g) \frac{d}{dt} + (uG(g))_x + (p - \bar{p}) \mathbf{u}_x = 0
\]

\[
\begin{array}{c}
g \leftrightarrow u = 0 \text{ at } t = \infty \\
\frac{d}{dt} \int G(g) \, dx + \int (p - \bar{p}) \mathbf{u}_x \, dx = 0
\end{array}
\]

\((5.1)\)
Now add (4.2) & (5.1) to get (again using \( u = 0 \) at \( \pm \infty \))

\[
\frac{d}{dt} \int_{-\infty}^{\infty} \left[ \frac{1}{2} u^2 + G(x) \right] dx + \varepsilon \int_{-\infty}^{\infty} u_x^2 dx = 0 \tag{6.1}
\]

On \( t \to 0 \):

\[
\frac{d}{dt} \int_{-\infty}^{\infty} \left[ \frac{1}{2} u^2 + G(x) \right] dx + \varepsilon \int_{-\infty}^{\infty} u_x^2 dx dl = \int_{-\infty}^{\infty} \frac{u_x^2}{2} + G(x) dx \tag{6.2}
\]

Dep. only on data

\( = C_0 \)

Note: the two bounds (3.1) & (6.2) are direct consequences of the equations, and they are integral estimates. Next comes a more subtle issue:

C. Pointwise bounds for \( g \).

Lemma: \( \exists \) const. \( C \) (depending only on \( g_0, u_0 \)) such that the following holds:

Given any \((x_0, t_0) \in \mathbb{R} \times [0, \infty)\), \( \exists \bar{x} \in \mathbb{R} \) s.t.

\[
|\bar{x} - x_0| \leq C \quad \text{and} \quad \frac{1}{C} \leq g(\bar{x}, t_0) \leq C.
\]
\textbf{Pf.:} Under the assump's \( I \neq 0 \) we have:

\[ G(g) \geq 0 \quad \text{for} \quad g \geq 0, \quad G(0) = 0 \]

\[ G'(g) = \frac{g}{s} - \frac{P(g) - P(0)}{s} \]

\[ G''(g) = \frac{1}{s} \left( \frac{g}{s} + \frac{P(g) - P(0)}{s} \right)^2 - \frac{G(g)}{s^2} + \frac{P(g)}{s^2} - \frac{P(0)}{s^2} \]

\[ = \frac{P'(g)}{s} \to 0 \]

So: \( G \) is positive, strictly convex, w/a unique root at \( \bar{g} \):

[Example: if \( P(g) = g \), then \( G(g) = g \log g + 1 - g \).]

Next, given \( s > 0 \) let \( g_1 \& g_2 \) be s.t.: \( g_1 < \bar{g} \)

\[ G(g_1) = G(g_2) = s \]
With $C_0$ as in (6.2) we let $\varepsilon > C_0/\delta$ and we define the set $E_{\pm}(t)$ as:

$$E_{\pm}(t) = \{ x \in \mathbb{R} \mid |x-x_0| < \varepsilon \land g(x,t) \leq g_1 \}$$

Clearly:

$$C_0 \geq \int_{E_{\pm}(t)} g(x,t) \, dx \geq \frac{\int_{E_{\pm}(t)} g(x,t) \, dx}{E_{\pm}(t)} \geq \frac{\delta}{\delta} E_{\pm}(t)$$

$$\implies |E_{\pm}(t)| \leq \frac{C_0}{\delta} \implies |E_{\pm}(t) \cup E_{\pm}(t)| \leq 2C_0 < \infty,$$

by choice of $\varepsilon$.

Thus: Given any $(x_0,t_0)$, if we consider the interval $[x_0-\varepsilon,x_0+\varepsilon]$, then there must be points in it which are not in $E_{\pm}(t_0) \cup E_{\pm}(t_0)$.

In other words: $\exists x \in \mathbb{R} \s.t. g_1 < g(x,t_0) < g_0$.

The claim is therefore satisfied by $C := \max \{ \varepsilon, g_2, g_1^{-1} \}$.\[\text{Q.E.D.}\]
With this we can proceed to estimate $\psi$:

Fix $t_0$ and choose $\bar{x}$ as in Lemma: $|x_0 - x_0| < C$

$\frac{1}{C} \leq s(x, t) \leq C$.

Let $x, x_0$ be the particle trajectories passing through $x$ & $x_0$, resp., at $t_0$:

\[
\begin{align*}
\frac{dx}{dt} &= u(x(t), t) \quad x(t_0) = x_0 \\
\frac{dx_0}{dt} &= u(x_0(t), t) \quad x_0(t_0) = x_0
\end{align*}
\]

Assume $x(t) > x_0(t)$ & compute:

\[
\frac{dx}{dt} (x(t) - x_0(t)) = \frac{\partial u(x(t), t)}{\partial x} - \int_{x_0(t)}^{x(t)} u_x(x, t) \, dx
\]

Cauchy-Schwarz:

\[
\begin{align*}
\left( \int_{x_0(t)}^{x(t)} u^2(x, t) \, dx \right)^{1/2} &\leq \int_{x_0(t)}^{x(t)} \left( u^2(x, t) + u_x(x, t) \right) \, dx \\
\frac{dx}{dt} (x(t) - x_0(t)) &\geq -\frac{1}{2} \left( \int_{x_0(t)}^{x(t)} \left( u_x(x, t) \right)^2 \, dx \right)^{1/2} + \int_{x_0(t)}^{x(t)} u_x(x, t) \, dx
\end{align*}
\]

With $a := x(t) - x_0(t)$ we thus have:

\[
\begin{align*}
a + \frac{1}{2} a &\geq -\frac{1}{2} \int_{x_0(t)}^{x(t)} u_x^2 \, dx \\
&\geq -\frac{1}{2} \int_{x_0(t)}^{x(t)} u_x^2 \, dx
\end{align*}
\]

For $\frac{1}{t} < \frac{1}{t_0}$:

\[
a(t) \frac{dx}{dt} \leq a(t) + \frac{1}{2} \int_{x_0(t)}^{x(t)} u_x^2 \, dx + \frac{t_0}{2} \int_{x_0(t)}^{x(t)} u_x^2 \, dx dt
\]
Thus: if $X(t) > X_0(t)$, then for $t \in [0, t_0]$, we have

$$0 < X(t) - X_0(t) \leq (X(t_0) - X_0(t_0)) e^{\frac{t-t_0}{2}} + \int_{t_0}^{t} e^{\frac{t-t'}{2}} u_x \, dx \, dt$$

$$< (X(t_0) - X_0(t_0)) e^{\frac{t}{2}} + C_0 \frac{1}{2} e^{\frac{t}{2}}, \quad \text{by (6.2)}.$$ 

Thus: given $t_0$, $\exists C(t_0)$ s.t. $\forall t \in [0, t_0]$ there holds

$$|X(t) - X_0(t)| \leq C(t). \quad (11.1)$$

(The argument in case $X(t) < X_0(t)$ is entirely similar. Note: $C(t)$ depends on data.)

Let's now compare $L = \log g$ at $(x_{t_0}, t_0)$ & $(x_0, t_0)$. We have:

$$\frac{d}{dt} L(X(t), t) = \frac{\delta_l}{3} + \delta_u \cdot u = -u_x(X(t), t), \quad \text{so:}$$

$$\frac{d}{dt} L(X_0(t), t) = -\int_{x_0}^{x(t)} \frac{u_{xx}}{Z_0(t)} \, dx \quad \text{from (11.2)}$$

$$\frac{d}{dt} L \left|_{x(t)} \right. = -\int_{x_0(t)}^{x_0(t)} \left[ (\delta_l) + (\delta_u)^2 \right] \, dx \quad \text{from (11.2)}$$

s.t.: $\varepsilon \frac{d}{dt} L \left|_{x_0(t)} \right. = -\int_{x_0(t)}^{x_0(t)} \left[ (\delta_l) + (\delta_u)^2 \right] \, dx \quad \text{from (11.2)}$

$$\varepsilon \frac{d}{dt} L \left|_{x(t)} \right. + \frac{P}{Z_0(t)} \left|_{x(t)} \right. = -\int_{x_0(t)}^{x_0(t)} \delta_u \, dx \quad \text{from (11.2)}$$

Next: define $\alpha(t) = P\left( g(X(t), t) \right) - P\left( g(X_0(t), t) \right)$, & let

$$\log g(X(t), t) = \log P(g(X(t), t)) - \log \left( P\left( g(X_0(t), t) \right) \right)$$

denominator is $\Delta$.
(11.2) Then reads:

$$\varepsilon \frac{d}{dt} (\Delta L) + \alpha : \Delta L = - \tilde{I}(t)$$  \hspace{1cm} (12.1)$$

where

$$I(t) := \int_{\mathbb{R}} (\varepsilon u(x,t)) dx$$  \hspace{1cm} (12.2)$$

Observe that:  \hspace{1cm} \alpha(t) = \frac{P(y_1) - P(y_2)}{\log y_1 - \log y_2} > 0 \hspace{1cm} \text{since} \hspace{0.2cm} I' > 0.$$

Thus: integrating from $t = 0$ to $t < t_0$ yields:

$$\Delta L(t) = e^{-\int_0^t \frac{\alpha(s)}{\varepsilon} ds} \cdot \Delta L(0) - \frac{1}{\varepsilon} \int_0^t e^{-\int_0^s \frac{\alpha(r)}{\varepsilon} dr} \tilde{I}(s) ds$$

As $\alpha(t) > 0$ \hspace{1cm} \text{we get that:} \hspace{1cm} 

$$|\Delta L(t)| \leq |\Delta L(0)| + \frac{1}{\varepsilon} |I(t)| + \frac{1}{\varepsilon} |I(0)| + \frac{1}{\varepsilon^2} \int_0^t \alpha(s) e^{-\int_0^s \frac{\alpha(r)}{\varepsilon} dr} I(s) ds$$

Recalling (11.1), estimate $A_1$ and $|\tilde{R}(x)| < C_0$, we get for $t < t_0$:

$$\int_{\mathbb{R}} \tilde{R} e^{(x,t)} dx = \int_{\mathbb{R}} \tilde{R} e^{x(t)} dx \leq C_0 \cdot |\tilde{R}_0 - \tilde{R}_0(0)| \leq C(t_0).$$  \hspace{1cm} (12.3)$$
By Cauchy-Schwarz & estimate B (6.2) we thus have:

$$|I| = \int_{X_0}^X g u \, dx \leq \left( \int_{X_0}^X g^2 \, dx \right)^{1/2} \left( \int_{X_0}^X u^2 \, dx \right)^{1/2} \leq C(t_0).$$

Thus:

$$|A_L(t)| \leq \left| C(t_0) + C(t_0) \int_0^t e^{-\frac{t-s}{t-t_0}} \, ds \right|$$

$$= C(t_0) \left[ e^{-\frac{t}{2}} \right]_{s=0}^{s=t}$$

$$\leq C(t_0), \text{ depending only on data } \& \, t_0.$$ 

Recalling what $A_L$ is we get for $t < t_0$:

$$\left| \log g(X(t), t) - \log g(X_0(t), t) \right| \leq C(t_0)$$

$$\Rightarrow$$

$$\left| \log g(X(t), t) \right| \leq \left| \log g(X(t), t) \right| + C(t_0)$$

Finally, apply the lemma: given $(x, t) \in \mathbb{R} \times [0, \infty)$, $\exists \, \alpha \in \mathbb{R}$ s.t.

$$|X_0 - X_0| \leq C \; \text{ & } \; \frac{1}{C} \leq g(x_{t, 0}) \leq C \Rightarrow \frac{1}{C} \leq \left| \log g(x_{t, 0}) \right| \leq C.$$ 

So: let the particle paths $X(t) \& X_0(t)$ be those passing through $x_0 \& x_0$, resp. at time $t_0$. We conclude that $E = E(t_0, \text{data})$.

$$\frac{1}{C} \leq g(x_{0, t_0}) \leq C.$$  \hfill (1.3, 1)
Simpler case: Bounded domain.

Consider the IBVP:

\[
\begin{align*}
\frac{\partial u}{\partial t} + (gu)_x &= 0, \\
(3u) + (gu^2 + p(x)) &= 0.
\end{align*}
\]  

(\text{NS})

with \(x\) in a bounded interval \([a, b]\) \((a, b < \infty)\) w/ no-slip boundary conditions: \(u(a, t) = u(b, t) = 0\), and initial data \(u_0\).

As above we assume that \(\exists \, \epsilon > 0\):

\[
\frac{1}{C} < g(x) \leq C \text{ for } \frac{1}{2} \leq x \leq \frac{3}{2}, \quad \text{and that } u_0 \in L^\infty([a, b]).
\]

Again we consider a (suff.) smooth soln. \((u, v)\) for which we seek a priori estimates.

Estimate A: is clearly still valid, but it will not be of much use now (since we automatically know that particle traj's stay "close" due to the finite length of our domain).
* Estimate B: Exactly the same argument as on p. 5

(except that x-integrals are now over [a, b] instead of \((\infty, \infty)\))

shows that: (provided \(uu_x \rightarrow 0\) at \(a, b\))

\[
\frac{d}{dt} \left( \int_a^b \frac{1}{2} u_x^2 \, dx \right) + \int_a^b u_{xx} \, dx + \varepsilon \int_a^b u_x^2 \, dx = 0. \tag{15.1}
\]

Similarly, with \(\bar{p} \neq 0\) chosen as an arbitrary fixed constant, we get that

\[
\frac{d}{dt} \left( \int_a^b G(p) \, dx \right) + \int_a^b (p - \bar{p}) u_x \, dx = 0 \tag{15.2}
\]

Adding (15.1) & (15.2), using that \(\bar{p}_x = 0\) & that \(u = 0\) at \(a, b\), we get the energy estimate:

\[
\int_a^b \frac{1}{2} u_x^2 + G(p) \, dx + \varepsilon \int_a^b u_x^2 \, dx \, dt \leq C_0 (\text{on initial data}) \tag{15.3}
\]
Estimate $C$: The proof of the Lemma on p. 7 goes through unchanged. We thus know that for whatever point we consider there are nearby points with high densities (above & away from 0, and in terms of both only).

The main simplification is that we can now skip the argument leading to (11.1) & instead proceed directly w/the estimate for $\Delta L(t) = \log g(X(t), t) - \log g(X_0(t), t)$, where

$X(t)$ & $X_0(t)$ are the path trajectories through two points $(X_0(t_0), t_0)$ & $(X_0(t_0), t_0)$, respectively. (For now $x_0$, $t_0$ are arbitrary.)

As before (pp. 11-12.) we get:

$$\Delta L(t) = e^{-J(x_0,t)} \Delta L(t_0) - \frac{1}{2} \left\{ I(t) - e^{-J(x_0,t)} I(t_0) - \frac{1}{2} \int_{t_0}^{t_0} \alpha(s) \frac{d}{ds} e^{-J(x_0,s)} I(s) ds \right\}$$

where $J(x_0,t) = \int_{t_0}^{t} \alpha(s) ds$ & $I(s) = \int_{\mathbb{R}} (x_0(x,s) dx, \alpha(s) \frac{d}{ds} e^{-J(x_0,s)} I(s) ds$, given by (11.3).
As before we get:

$$|\Delta L(t)| \leq |\Delta L(0)| + \frac{1}{\varepsilon} |I(t)| + \frac{1}{\varepsilon^2} \int_0^t x(s) \int_0^s I(u) \, du \, ds.$$ 

However, now we immediately have:

$$|I(0)| = \left| \int_{X_0(0)}^{X(0)} p(x, s) u(x, s) \, dx \right| \leq \left( \int_{X_0(0)}^{X(0)} p(x, s) \, dx \right)^{\frac{1}{2}} \left( \int_{X_0(0)}^{X(0)} u^2 \, dx \right)^{\frac{1}{2}}$$

$$\leq C_0 \left( \int_{X_0(0)}^{X(0)} p(x) \, dx \right)^{\frac{1}{2}} \leq C_0 \cdot C \, |X(0) - X_0(0)|$$

$$\leq C_0 \cdot C \cdot (t - a) \leq C_0 + \frac{\text{Dep. only on data}}{\text{on data}}$$

As on p. 13, we thus get:

$$|\log g(X_0(t), t)| \leq |\log g(X(t), t)| + C.$$ 

The final part of the argument is as before:

Given $(x_0, t_0) \in [a, b] \times [0, a]$, pick $x_0$ s.t. $(|x_0 - x_0| \leq C)$

$$\frac{1}{C} \leq g(x_0, t_0) \leq C,$$

where $C$ only depends on data.

Applying the argument above for these $x_0, x_0$ & $t_0$, we conclude

that $\exists C$ dep. only on the data s.t. $\frac{1}{C} \leq g(x_0, t_0) \leq C.$
Now, let’s return to the Cauchy problem posed on all of \( \mathbb{R} \). Before proceeding we note a useful consequence of what we have established so far. We know that for \( T > 0 \) given there is a const. \( C \) dep. on \( g_0, u_0 \) & \( T \), s.t. \( \frac{1}{C} < g(x,t) < C \quad \forall x, t \in I \).

That is: all \( g \)-values up to time \( T \) is in some finite interval \( [\frac{1}{C}, C] \), and \( G(g) \) is strictly convex there:

Thus, given any other positive, smooth, func. \( H(g) \sim H(\bar{g}) = H'() = 0 \) &

with \( H''(\bar{g}) < \infty \), there exists a sufficiently small \( \delta > 0 \) s.t.

\[
\delta \cdot H(g) \leq G(g) \quad \forall g \in [\frac{1}{C}, C].
\]

Moreover:

Thus, for any such \( \delta \) const. \( \bar{g} \) dep. on \( T, g_0, u_0 \) s.t.

\[
\sup_{x \in I} \int_{x_0}^{x_1} H(g(x,t)) \, dx \leq \bar{g},
\]

We will use this in the last estimate, w/ \( H(g) = (P(g) - P(\bar{g}))^2 \).

Also, recall that \( \forall \delta > 0 \) we have:

\[
\forall \delta \leq \frac{\delta^2}{2} + \frac{1}{2} \delta^2 \quad \forall \delta \in \mathbb{R}.
\]
$D. H^1$ estimates

Our first goal is to prove the following: Given $T > 0$, there exists a constant $C(T)$ (dep. on $g_0, u_0, T$) such that:

$$\sup_{t \in [0, T]} \int_{\mathbb{R}} |u(x, t)|^2 dx + \int_0^T \int_{\mathbb{R}} |u_t(x, t)|^2 dx dt \leq C(T)$$

(19.1)

Here $G(t) = \min(g, 0, T) = 1$ at $t = G(t) \rightarrow \infty$ 

The proof is based on a careful analysis of the momentum eq.

**Notation:** In what follows $\overline{\int}$ will denote $\int_0^T \int_{\mathbb{R}}$ $dx dt$, and if nothing else is explicitly indicated, all $x$-integrals are over $\mathbb{R}$ & all $t$-integrals over $[0, T]$.

The momentum eq. is given as:

$$g u + \int_{\mathbb{R}} = \varepsilon u_{xx}$$

(19.2)

We multiply by $5u$ & integrate in $x \& t$:

$$\overline{\int 5u^2} + \overline{\int u P_x} = \varepsilon \overline{\int 5u u_{xx}}$$

(19.3)
Substituting in (1.3) gives:

\[ 2 \int_0^1 \int_0^x (x-y)^2 \, dy \, dx. \]

(Note that we have used properties \( \mathcal{F} = 0 \) in the works above.)

\[ - \int_0^1 \int_0^x (x-y)^2 \, dy \, dx = \int_0^1 x^2 - 2x \, dx = \frac{1}{3}. \]
\[
\int_0^T \frac{\partial^2 u}{\partial t^2} \, dx + \int_0^T \frac{\partial^2 t}{\partial x^2} \, dx = \frac{\partial}{\partial t} \int_0^T \int_0^1 u_x \, dx \, dt - \frac{\partial}{\partial x} \int_0^T \int_0^1 u_t \, dx \, dt
\]

The next step is to bound/absorb into the L.H.S. the terms (c) - (g):

\[|c| \leq C(T) \text{ by (6.2)} \]

\[\hbox{\bf (d): latter! (this is the hard one) } \]

\[|e| \leq \frac{\varepsilon^2}{2} \int_0^1 u_x^2 \, dx + \int_0^T \frac{\partial}{\partial t} (|E^2|) \, dt \leq \frac{C(T)}{2} \text{ by (18.2)} \]

\[\leq \varepsilon \int_0^1 u_x^2 \, dx + \frac{1}{2 \delta} \int_0^T |E|^2 \, dt \leq (13.1) \text{ by (18.1) (\#(e) = (18.2) - (13.1))} \]

We'll choose \( \delta \) small & absorb the first term into the L.H.S.

\[\hbox{\bf (14.1) } \int_0^1 u_t^2 \, dx + \int_0^T (|E|^2) \, dt \leq C(T) + \int_0^T |E| \, dt \leq C(T) \text{ (by (14.1)). } \]

\[\hbox{\bf (15.1) } \int_0^1 u_x^2 \, dx \leq C(T) \]

Choosing \( \delta = \frac{T}{2} \), say, & substituting these into (21.1) we get:
\[ S S E \bar{u}^2 + \frac{3}{4} E(x) S u_x^2 dx \leq C(T) + \varepsilon S S E \frac{1}{u_x^2} \]  \hspace{1cm} (22.1)

Note: the L.H.S. is (essentially) the L.H.S. of (19.1). Thus we only need to bound the \( \varepsilon \) term. We could try:

\[ S S E \varepsilon u_x^2 \leq \varepsilon \int \left| u_x(x, t) \right|^4 \left( \int S u_x^2 dx \right) dx \]  \hspace{1cm} (22.2)

and if we could bound \( u_x(x, t) \) we would be done estimating \( \varepsilon \) (by (6.2)). We could try to write:

\[ u_x(x, t)^2 = 2 \int u_x(x, t) u_x dx \leq C \left( \int S u_x^2 dx \right)^{1/2} \left( \int S u_{xx}^2 dx \right)^{1/2} \]

but the last integral is not available ...

This is a subtle issue & requires a more detailed analysis.

First of all let's define the 'effective viscous flux' \( F \) by:

\[ F := E u_x - (P(x) - \frac{\varepsilon}{u_x^2}) \]  \hspace{1cm} (P = P(x) = const.)

such that the momentum eq. takes the form:

\[ \ddot{u} = F_x \]
We have:

\[ F(x,t) = \int_0^t \int_{\mathbb{R}^n} \left| \nabla u \right|^2 \, dx \, dt \leq C \left( \int \left( \nabla u \right)^2 \, dx \right)^{1/2} \left( \int \left| u \right|^2 \, dx \right)^{1/2} \]

\[ (a+b)^2 \leq 2(a^2 + b^2) \]

\[ \leq C \left( \int \left( \nabla u \right)^2 \, dx \right)^{1/2} \left( \int \left| u \right|^2 \, dx \right)^{1/2} \]

\[ \text{(12.1) & (13.1)} \]

\[ \leq C \left[ 1 + \left( \int \left( \nabla u \right)^2 \, dx \right)^{1/2} \right] \left( \int \left| u \right|^2 \, dx \right)^{1/2} \]

By (13.1) we thus get from \( u_y = \varphi^{-1} (\varphi - \bar{\varphi}) + F \) that:

\[ \left| u_y \right| \leq C \left[ 1 + \left| F \right| \right] \leq C \left\{ 1 + \left[ 1 + \left( \int \left( \nabla u \right)^2 \, dx \right)^{1/2} \right]^2 \left( \int \left| u \right|^2 \, dx \right)^{1/2} \right\} \]

\[ \leq C \left\{ 1 + \left[ 1 + \left( \int \left( \nabla u \right)^2 \, dx \right)^{1/2} \right]^2 \left( \int \left| u \right|^2 \, dx \right)^{1/2} \right\} \]

\[ \text{(13.1)} \]

With this we can proceed to estimate the \( \varphi \)-term. By (2.2):

\[ \int \int 5|u_y|^3 \leq \int \int 5(\varphi - \bar{\varphi}) \int_0^\infty \left( \int \left( \nabla u \right)^2 \, dx \right) \, dt \]

\[ \leq C \int \int 5(\varphi - \bar{\varphi}) \left\{ 1 + \left[ 1 + \left( \int \left( \nabla u \right)^2 \, dx \right)^{1/2} \right]^2 \left( \int \left| u \right|^2 \, dx \right)^{1/2} \right\} \left( \int \left( \nabla u \right)^2 \, dx \right)^{1/2} \left( \int \left| u \right|^2 \, dx \right)^{1/2} \, dt \]

\[ \leq C \int \int \left[ \int \left( \nabla u \right)^2 \, dx \right] \left( \int \left| u \right|^2 \, dx \right) \, dt + \int \int \left[ 5(\varphi - \bar{\varphi}) \right] \left( \int \left( \nabla u \right)^2 \, dx \right)^{1/2} \left( \int \left| u \right|^2 \, dx \right)^{1/2} \, dt \]

\[ \leq C \int \int \left( \int \left( \nabla u \right)^2 \, dx \right) \left( \int \left| u \right|^2 \, dx \right) \, dt + \int \int \left[ 5(\varphi - \bar{\varphi}) \right] \left( \int \left( \nabla u \right)^2 \, dx \right)^{1/2} \left( \int \left| u \right|^2 \, dx \right)^{1/2} \, dt \]

\[ (a \leq C [1 + t^{2/3}]) \]

\[ \leq C \int \int \left( \int \left( \nabla u \right)^2 \, dx \right) \left( \int \left| u \right|^2 \, dx \right) \, dt + \int \int \left[ 5(\varphi - \bar{\varphi}) \right] \left( \int \left( \nabla u \right)^2 \, dx \right)^{1/2} \left( \int \left| u \right|^2 \, dx \right)^{1/2} \, dt \]
\[
\left(\text{denote } w^{1
\text{ (exponents } 4 \& \frac{4}{3} \text{ in 1st term)} }\right) \leq C(T) \left\{ 1 + \left( \int E(t) u_x^2 \, dx \right)^{\frac{3}{4}} + \sup_{\alpha \leq t \leq T} \left( E(t) \int u_x^2 \, dx \right)^{\frac{3}{4}} \left( \int u_x^2 \, dx \right)^{\frac{1}{4}} \right\}^{\frac{3}{4}}.
\]

Summing up we thus get from (22.1) that, using the fact that \( T \) is arbitrary, lower bound on \( g \) in 2nd term on L.H.S.,

\[
\sup_{\alpha \leq t \leq T} E(t) \int u_x^2 \, dx + \int u_x^2 \right\} \leq C(T) \left\{ 1 + \left( \int E(t) u_x^2 \, dx \right)^{\frac{3}{4}} + \sup_{\alpha \leq t \leq T} \left( E(t) \int u_x^2 \, dx \right)^{\frac{3}{4}} \right\}^{\frac{3}{4}}.
\]

Doubling the L.H.S. by \( \tilde{t} \) we thus have:

\[
\tilde{t} \leq C(T) \left\{ 1 + \tilde{t}^{\frac{3}{4}} \left( 1 + \tilde{t}^{\frac{3}{4}} \right)^{\frac{3}{4}} \right\} \leq C(T) \left\{ 1 + \tilde{t}^{\frac{3}{4}} \right\} \leq C(T) \left\{ 1 + \tilde{t}^{\frac{3}{4}} \right\},
\]

so that necessarily \( \tilde{t} \leq C(T) \), which concludes the proof of the claim on p. 19.
We finish off by stating the following higher order estimate:

Given \( T > 0 \), then there exists a constant \( C(T) \) (depending only on \( T, \sigma_a \& u_0 \)) s.t.

\[
\sup_{0 \leq t \leq T} \left\{ \frac{1}{2} \int \|\dot{u}(x,t)\|^2 \, dx \right\} + \int_0^T \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(t, x) \left( \frac{\partial u}{\partial x}\right)^2 \, dx \, dt \leq C(T). \tag{25.1}
\]

\[\text{Pf: on pp. 26-29.}\]
2nd Higher Order Energy Estimate:

Momentum eq: \( \dot{p} + P_x = \varepsilon u_{xx} \) \hspace{1cm} (26.1)

Take material derivative: \( \dot{g}u + g\ddot{u} + (P_x) = \varepsilon (u_{xx}) \)

\( \rightarrow \quad -\rho u_x \dot{u} + g\ddot{u} + P_{xt} + u \dot{P}_{xx} = \varepsilon (u_{xxt} + uu_{xxx}) \) \hspace{1cm} (26.2)

\( \rightarrow \quad -u_x (-P_x + \varepsilon u_{xx}) + g\ddot{u} + P_{xt} + u \dot{P}_{xx} = \varepsilon u_{xxt} + uu_{xxx} \)

\( \rightarrow \quad g\ddot{u} + P_{xt} + (P_x)_x = \varepsilon (u_{xxt} + uu_{xxx}) \)

Multiply by \( 5(s)^2 u \) & integrate in \( x \) & \( t \) & rearrange:

\[ \int \int 5(s)^2 u \ddot{u} \, dx \, dt = \int \int 5(s)^2 u \left[ -P_{xt} - (P_x)_x + \varepsilon (u_{xxt} + uu_{xxx}) \right] \, dx \, dt \] \hspace{1cm} (26.3)

LHS = \[ \int \int 5(s)^2 \left[ \dot{u}(\dot{u}) + uu(\dot{u})_x \right] \, dx \, dt \]

\[ = \int \int 5(s)^2 \frac{(\dot{u})^2}{2} \, dx \, dt + \int \int 5(s)^2 uu(\dot{u})_x \, dx \, dt \]

\[ = \int \int \left[ 5(s)^2 \frac{(\dot{u})^2}{2} \right] \, dx \, dt - \int \int \left( 5(s)^2 \frac{(\dot{u})^2}{2} + uu(\dot{u})_x \right) \, dx \, dt \]

\[ = \frac{5(s)^2}{2} \int \int (\dot{u})^2 \, dx \, dt - \int \int \left( 5(s)^2 (\dot{u} + uu) \right) \, dx \, dt \]

\[ = \frac{5(s)^2}{2} \int \int (\dot{u})^2 \, dx - \int \int \left( 5(s)^2 (\dot{u} + uu)_x \right) \, dx \, dt \] \hspace{1cm} (26.4)
The ε-term on the R.H.S of \((26.3)\) may be rewritten as follows:

\[
\iint \varepsilon (t^2 u_x^2 + (u u_{xx})_x) = -\iint \varepsilon (t^2 u_x) (u_{xx} + uu_{xx}) = \iint \varepsilon (u_x) (u_x^2 - u_x^2)
\]

Now: \((\dot{a}_x) = (\dot{a}_x) - u_x a_x\), so that

\[
\iint \varepsilon (t^2 u_x) (u_{xx} + uu_{xx}) = -\iint \varepsilon (t^2) \left[ (u_x^2 - u_x^2) \right]
\]

\[
= -\iint \varepsilon (t^2) (u_x^2) + \iint \varepsilon (t^2) u_x^2 (u_x^2)
\]

Substituting (27.1) \& (26.7) into (26.3) then yields:

\[
\frac{\varepsilon (T)}{2} \iint \xi |u_x|^2 d\xi + \varepsilon \iint \varepsilon (t^2) (u_x^2) = \varepsilon \iint \varepsilon (t^2) u_x^2 (u_x^2) + \iint \varepsilon (t^2) u_x^2 (u_x^2)
\]

\[
- \iint \varepsilon (t^2) u_x \left[ p_{xx} + (p u)_x \right]
\]

For the last term in (27.2):

\[
\left| \iint \varepsilon (t^2) u_x \left[ p_{xx} + (p u)_x \right] \right| = \left| -\iint \varepsilon (t^2) (u_x) \left[ p_{xx} + (p_u) u_x \right] \right|
\]

\[
= \left| \iint \varepsilon (t^2) (u_x^2) \frac{u_x}{2} \right| \leq C M \left\{ \frac{\varepsilon \iint \varepsilon (t^2) u_x^2}{2} + \frac{1}{2} \iint \varepsilon (t^2) u_x^2 \right\}
\]

(27.3)

Where we have used \( |a| \leq C |t| \).
For the first term on the R.H.S. of (32) we have:

$$
\varepsilon \int \int 5(4)^{2} u_{x}(x) \, dx \leq \varepsilon \left( \frac{5}{2} \int \int 5(4)^{2} u_{x}^{2}(x) \, dx + \frac{1}{28} \int \int 5(4)^{2} u_{x}^{4}(x) \, dx \right)
$$

(32.1)

Substituting (32.3) & (32.1) into (32.2), using 1st H-estimate, using lower bound in 1st term on the L.H.S. & energy est., yield:

$$
5(4)^{2} \int u_{x}^{2} \, dx + 8 \int 5(4)^{2} |u_{x}|^{2} \, dx \leq C(T) \left( 1 + \int \int 5(4)^{2} u_{x}^{4} \, dx \right)
$$

(32.2)

To handle the last term we argue as before:

$$
\int \int 5(4)^{2} u_{x}^{4} \, dx \leq \int \int 5(4)^{2} \|u_{x}(x,t)\|_{L_{x}}^{1} \cdot \left( \int |u_{x}|^{2} \, dx \right) \, dt
$$

Using (32.1) we have:

$$
|u_{x}|^{2} \leq C(T) \left\{ 1 + \left[ 1 + \left( \int u_{x}^{2} \, dx \right)^{1/2} \right] \cdot \left( \int |u_{x}|^{2} \, dx \right)^{1/2} \right\}^{2}
$$

such that:

$$
\int \int 5(4)^{2} u_{x}^{4} \, dx \leq C(T) \left\{ 1 + \left[ 1 + \left( \int u_{x}^{2} \, dx \right)^{1/2} \right] \cdot \left( \int |u_{x}|^{2} \, dx \right)^{1/2} \right\} \cdot \left( \int |u_{x}|^{2} \, dx \right)^{1/2} \cdot \left( \int |u_{x}|^{2} \, dx \right)^{1/2} \, dt
$$

By original & ar($u_{x}^{2}$)

$$
\leq C(T) \left\{ 1 + \left[ 1 + \left( \sup_{t \in [0,T]} \int u_{x}^{2} \, dx \right)^{1/2} \right] \cdot \left( \int |u_{x}|^{2} \, dx \right)^{1/2} \right\} \cdot \left( \int |u_{x}|^{2} \, dx \right)^{1/2} \, dt
$$

(1st H-estimate)

$$
\leq C(T) \left\{ 1 + \left( \sup_{t \in [0,T]} \int u_{x}^{2} \, dx \right)^{1/2} \right\} \cdot \left( \int |u_{x}|^{2} \, dx \right)^{1/2} \, dt
$$
Using that \((5(t) \int |u(x)|^2 dx)^{1/2} \leq C(T)\) by the 1st \(H^1\) estimate, and the fact that \(\int 5(t)^{-1/2} dt \leq C(T)\), we get:

\[
\int 5^2 |u(x)|^4 \leq C(T) \left\{ 1 + \left( \sup_{t \in [0,T]} \int |u(x)|^2 dx \right)^{1/2} \right\}^7
\]

Thus, bounding the L.H.S. at \((3.2)\) by \(B(T)\), we get:

\[
B(T) \leq C(T) \left\{ 1 + B(T)^{1/2} \right\}^7,
\]

so that necessarily, \(B(T) \leq C(T)\), which concludes the proof of the estimate on p. 25.
Consequences of higher order energy estimates:

Bound on maximal velocity:

\[ |u(x,t)|^2 = \frac{2}{\pi} \int u u_x \, dx \leq C \left( \int u_x^2 \, dx \right)^{1/2} \left( \int u^2 \, dx \right)^{1/2} \]

\[ (\Delta^{-\frac{1}{2}} C(t)) \leq C \left( \int u_x^2 \, dx \right)^{1/2} \left( \int u^2 \, dx \right)^{1/2} \]

Energy estimate

\[ \frac{3}{4} \int |u(\cdot, t)|^4 \, dx \leq C(t) \]

So:

\[ \left( \int u_x^2 \, dx \right)^2 \leq C(t) \]  \hspace{1cm} (3.0.1)

Next: since \( \dot{u} = u_t + uu_x \) we have

\[ \int \int \tilde{e}(t) u_x^2 \leq C \int \int \tilde{e} \left( u_t^2 + u_x^2 \right) \]

\[ \leq C(t) \left\{ 1 + \int 5(t) \|u(\cdot, t)\|_{L^2}^2 \left( \int u_x^2 \, dx \right) \, dt \right\} \]

\[ \leq C(t) \left\{ 1 + \int 5(t)^{1/2} \left( \int u_x^2 \, dx \right) \, dt \right\} \leq C(t), \]

where we have used the various energy estimates.

So:

\[ \int \int \tilde{e}(t) u_x^2 \leq C(t) \]  \hspace{1cm} (3.0.2)
We can now establish Hölder continuity in space & time:

**Space:**

Fix $0 < t < T$ & $x_i < x_e$:

$$
|u(x_{e}, t) - u(x_{i}, t)| \leq \int_{x_{i}}^{x_{e}} \left( \int_{x_{i}}^{x_{e}} |u| \ dx \right)^{2} \right)^{1/2} \leq \sqrt{x_{e} - x_{i}} \cdot \left( \int_{x_{i}}^{x_{e}} |u_x| \ dx \right)^{1/2}
$$

$$
\leq \sqrt{x_{e} - x_{i}} \cdot 6(t)^{1/2} \left( \int_{x_{i}}^{x_{e}} u_x^2 \ dx \right)^{1/2}
$$

$$
\leq C(T) \frac{\sqrt{x_{e} - x_{i}}}{\sqrt{T}}
$$

(3.1)

Set: $u(\cdot, t)$ is Hölder cont. w/ exponent $1/2$ for $t \in [T, T]$.

**Time:**

Fix $0 < t < T$, $t_{1} < t_{2} \leq T$, $h > 0$ & $x \in \mathbb{R}$:

$$
|u(x, t_{2}) - u(x, t_{1})| \leq \frac{1}{h} \int_{x}^{x_{t_{2}}} |u(x, t_{2}) - u(y, t_{2})| \ dy
$$

$$
+ \frac{1}{h} \int_{x_{t_{2}}}^{x} |u(y, t_{2}) - u(y, t_{1})| \ dy
$$

$$
+ \frac{1}{h} \int_{x_{t_{1}}}^{x_{t_{2}}} |u(y, t_{1}) - u(x, t_{1})| \ dy
$$

$$
\leq C(T, \tau) \int_{x_{t_{1}}}^{x_{t_{2}}} |u(y, \cdot) - u(x, \cdot)| \ dy + \frac{1}{h} \int_{t_{1}}^{t_{2}} \int_{x}^{x_{t_{2}}} |u_{x}| \ dx \ dy
$$
\[
\leq \frac{c(T,T)}{h} \cdot h^{3/2} + \frac{1}{h} \left( \frac{\int_{-T}^{T} \int_{x-T}^{x+T} dy \, dx}{\int_{-T}^{T} dx} \right)^{1/2}
\]

\[
\leq \frac{c(T,T)}{h} \cdot h^{3/2} + \frac{1}{h} \sqrt{h(2T-t)} \cdot c(T) \left( \int_{-T}^{T} \int_{x-T}^{x+T} dy \, dx \right)^{1/2}
\]

\[
\leq c(T,T) \left( h^{1/2} + \sqrt{\frac{t_{2}-t_{1}}{h}} \right)
\]

where we have used that

\[t_{2}-t_{1} > T \quad \& \quad (30,2).\]

Now let \( h = \sqrt[4]{t_{2} - t_{1}} \), s.t.

\[
|u(x,t_{1}) - u(x,t_{2})| \leq c(T,T) \left( t_{2} - t_{1} \right)^{1/4}
\]

i.e. \( u \) is Hölder continuous (w/ exponent \( 1/4 \)) in time.
We now have the necessary ingredients to complete the proof of global existence. However, this still requires a substantial amount of work and will only briefly outline the steps.

**First:** Given \( g_{0} \), \( u_{0} \) w/ \( C_{0}^{-1} \leq g_{0} \leq C_{0} \) & \( 1 \leq g_{0} \leq 1 \) \( \| \partial_{x} g_{0} \|_{2} \leq C_{0} \), \( \| u_{0} \|_{2} \leq C_{0} \),

let \( g_{5} \), \( u_{5} \) be smooth approximations satisfying

\[
C_{0}^{-1} \leq g_{5} \leq C_{0} , \quad \| g_{5} \|_{2} \leq C_{0} , \quad \| u_{5} \|_{2} \leq C_{0} ,
\]

for some (if necessary, larger value of) \( C_{0} \), independently of \( g_{0} \)

& s.t. \( (g_{5}, u_{5}) \in H^{1}(\mathbb{R}) \) \( \forall \delta > 0 \).

**Now,** for a fixed \( \delta > 0 \), it is a fact (proved e.g. by a contraction mapping argument) that there exists a smooth \((H^{1})\) soln. \( (g_{\delta}, u_{\delta}) \) of the Navier-Stokes eqn. w/ data \( (g_{\delta}, u_{\delta}) \) on some short time interval \([0, T_{\delta}]\). Furthermore, it is a fact that the time \( T_{\delta} \) only depends on the \( H^{1} \)-norm of \( (g_{\delta}, u_{\delta}) \).
- We can now invoke our analysis to conclude that this smooth solution satisfies a series of a priori bounds. Among these are \( \delta \)-independent bounds on the \( H^1 \)-norm of \((g^\delta, u^\delta)\) at time \( t = T_1 \). (Precisely, these bounds only depend on \( \sup_{a_1} |a_1|, \sup_{a_2} u^a, ||u^1||_{L^2}, ||u^2||_{L^2} \), and the time \( T_1 \).) Note: we haven't done all of these, e.g. \( \delta \) has not been estimated, but this can be done with the available bounds.

- By the local existence of \( H^1 \)-solutions we can thus extend the solution from \( t = T_1 \) to \( t = T_1 + \bar{T} \), where \( \bar{T} \) only depends on the \( H^1 \)-norm of \((g^\delta, u^\delta)\) at \( t = T_1 \). That is:

  \[ \bar{T} \text{ is independent of } \delta. \]

- We can now use our a priori bounds again for \([0, T_1 + \bar{T}]\) and extend the solution to \([0, T_1 + 2\bar{T}]\). Again: the solution will
satisfy $\mathcal{S}$-indep. bounds for its $H^1$-norm. We can then extend to $0, T, +3\pi, \ldots$. In this way we reach $T$ in finitely many steps.

- So $(g^5, u^5)$ is a smooth soln. on all of $[0, T]$ & hence satisfies all of the $S$-independent (but $g_0, u_0, T$) dependent a priori bounds.

- This concludes the outline of global existence of smooth $(H^4)$ soln's for the Navier-Stokes eqs. One can now build upon this result & extend the class of data to $L^3$ (or upper & lower bounds) for $g_0$ & $H^2$ for $u_0$, as in the statement on p. 2. This part is more technical and we refer to Hoff, Lions, ... for details.