Given that the simplex method leads from the following input tableau \( M_0 \) (for which \( x = 0 \)) to the following terminal tableau \( M_r \):

\[
M_0 = \begin{bmatrix}
1 & x_1 & x_2 & x_3 \\
1 & 1 & 2 & 2 \\
1 & 2 & 2 & 2
\end{bmatrix}
\]

\[
M_r = \begin{bmatrix}
x_1 & x_2 & x_3 & 0 & 0 & 0 & 0 \\
1 & 0 & 4 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(a) determine the matrix \( Q_0 \) for which \( M_0 = Q_0 M_r \).

\[ Q_0 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \]

Since \( Q_0 M_0 = M_r \) and hence \( Q_0 M_0 = M_r \), we have \( M_0 = Q_0 M_r \). 

(b) do a post-optimality analysis assuming that the input constraint coefficient matrix and terminal basic sequence remain fixed — by giving explicit formulas in terms of \( b_1, b_2, c_1, c_2, c_3 \) and \( d \), of:

\[
\begin{align*}
x_1^* &= 0 \\
x_2^* &= 0 \\
x_3^* &= 0 \\
x_1^* &= 0
\end{align*}
\]

which are valid provided that

\[
\begin{align*}
\text{feasibility:} & \quad 3b_1 + 2b_2 + 0 \geq 0 \\
\text{ranges:} & \quad 3b_1 + 2b_2 \geq 0 \\
\text{optimality:} & \quad c_1 + c_2 - 4c_3 + 0 \geq 0 \\
\text{ranges:} & \quad c_1 + c_2 - 4c_3 \geq 0
\end{align*}
\]

(c) determine each of the following partial derivatives that definitely exist, and mark with an \( x \) those that might not exist:

\[
\begin{align*}
p_1 &= \frac{\partial f}{\partial x_1} = 1 \\
p_2 &= \frac{\partial f}{\partial x_2} = 1 \\
p_3 &= \frac{\partial f}{\partial x_3} = 1 \\
p_4 &= \frac{\partial f}{\partial b_1} = 1 \\
p_5 &= \frac{\partial f}{\partial b_2} = 1 \\
p_6 &= \frac{\partial f}{\partial c_1} = 1 \\
p_7 &= \frac{\partial f}{\partial c_2} = 1 \\
p_8 &= \frac{\partial f}{\partial c_3} = 1
\end{align*}
\]

because

\[
\begin{align*}
b_1 &= 1 + 10x_1 + 10x_2 \\
b_2 &= 1 + 10x_1 + 10x_2 \\
x_1 &= 0 \\
x_2 &= 0
\end{align*}
\]

the boundary of the boundary.
2. Perform one iteration of the dual simplex algorithm on each of the following

(31) canonical schemas (either by giving an optimal value and solution, or by saying why there are no feasible solutions, or by producing a single new schema).

<table>
<thead>
<tr>
<th>x_1</th>
<th>x_2</th>
<th>z</th>
<th>f(min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
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</tr>
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Also, insert the appropriate dual labels in one of the three preceding primal schemas (your choice).

3. Given the following non-canonical

(27) primal-dual schema where x_1 and y_1 are restricted, but x_2 is unrestricted and η_1 is artificial

\[
\begin{align*}
A_1: & \quad 0 \leq x_1, \quad y_1 \\
A_2: & \quad 0 \leq y_2 \\
A_3: & \quad x_1 + y_1 + 2y_2 = 1
\end{align*}
\]

(a) classify y_1, y_2, η_1, η_2 as restricted, unrestricted or artificial:

- y_1 is unrestricted
- y_2 is restricted
- η_1 is artificial
- η_2 is restricted

(b) state the appropriate complementary slackness conditions (in their simplest form)

\[
x_1 = 0 \text{ or } x_2 = 0, \quad y_2 = 0 \text{ or } y_3 = 0
\]

(c) reduce the given schema to canonical form, including a statement of any equations that must be stored to eventually determine complete primal and dual optimal solutions.

\[
\begin{align*}
x_1 & \quad x_2 \\
\end{align*}
\]

Canonical form

\[
\begin{align*}
x_2 & \quad x_1 \\
\end{align*}
\]

with stored equations

\[
\begin{align*}
2 & = 0x_2 + 2x_1 \\
1 & = -1x_2 + 3x_1
\end{align*}
\]

that determine x_2^* and y_2^* from x_1^* and y_3^*. 

2. Perform one iteration of the dual simplex algorithm on each of the following

(31) canonical schemas (either by giving an optimal value and solution, or by saying why there are no feasible solutions, or by producing a single new schema).

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\]

(c) reduce the given schema to canonical form, including a statement of any equations that must be stored to eventually determine complete primal and dual optimal solutions.
Given the following pair of primal and dual linear optimization problems P and Q:

**Problem P:** Minimize \( g(x) \) subject to
- \( Ax = b \)
- \( x \) unrestricted

**Problem Q:** Minimize \( yb \) subject to
- \( yA = 0 \)
- \( y \) unrestricted

(a) which, if any, of these two problems P and Q are always consistent (for all compatible matrices A, b, and \( y \))

**Problem Q, because** \( y = 0 \) is clearly feasible

(b) which, if any, of these two problems P and Q are always bounded when consistent

**Problem P, because** **Problem Q is always consistent**

(c) since it is known from linear algebra that each "vector space" \( V \) can be represented as the "column space" for some matrix A (i.e., as the set \( \{ Ax \mid x \text{ unrestricted} \} \)), and since it is also known that its orthogonal complement \( V^\perp = \{ x \mid A \cdot x = 0 \} \), parts (a) and (b) along with our duality theory imply that problem Q is bounded if, and only if, b lies in \( V \)

So, \( b \) is in \( V \)