Handout on the Stationarity Condition

In this handout we explore the nature of the stationarity condition in univariate ARMA time series models.

This general class of models is defined as follows. If \( v_t \) is generated via an \( ARMA(p, q) \) model then it has the following representation,

\[
\theta(L)v_t = \phi(L)w_t
\]

where the AR polynomial is \( \theta(L) = 1 - \sum_{i=1}^{p} \theta_i L^i \), the MA polynomial is \( \phi(L) = 1 + \sum_{i=1}^{q} \phi_i L^i \), the AR and MA polynomials have no roots in common, and \( w_t \sim iid(0, \sigma^2) \).

Recall from class discussion that the stationarity condition is required to ensure that \( v_t \) also has a well defined \( MA(\infty) \) representation,

\[
v_t = \psi(L)w_t
\]

where \( \psi(L) = 1 + \sum_{i=1}^{\infty} \psi_i L^i \).

To motivate the nature of the stationarity condition in the general case, we begin by considering the nature of the condition in \( AR(1) \) and \( AR(2) \) models, and then state the general form of the condition for \( AR(p) \) models.

\( i \) The stationarity condition in the \( AR(1) \) model:

Consider the model

\[
v_t = \theta_1 v_{t-1} + w_t
\]

Recall from our class discussion that (3) implies that

\[
v_t = \lim_{k \to \infty} \left\{ \theta_1^k v_{t-k} + \sum_{i=0}^{k-1} \theta_1^i w_{t-k-i} \right\}
\]

and hence the condition for stationarity is that \( |\theta_1| < 1 \).

This condition can be equivalently stated in terms of the roots of the AR polynomial, \( \theta(L) = 1 - \theta_1 L \).

To this end, consider the equation \( \theta(\tilde{m}) = 0 \) for scalar \( \tilde{m} \), that is

\[
1 - \theta_1 \tilde{m} = 0
\]
The solution to (5) is \( \hat{m} = \theta_1^{-1} \). Therefore, the condition for stationarity can be equivalently expressed as the requirement that the root of the AR polynomial is greater than one in absolute value.

This alternative formulation of the condition is intimately tied to the stability condition for the first order linear homogeneous difference equation based on the AR polynomial, that is

\[
\theta(L)y_t = y_t - \theta_1 y_{t-1} = 0
\]

(6)

It can be shown that the general solution to (6) takes the form

\[
y_t = b(m)^t
\]

(7)

where \( m \) is the root to the equation

\[
m - \theta_1 = 0
\]

(8)

and \( b \) is a constant determined by the initial conditions \( (b = y_0) \). From (7), it can be seen that if \( |m| < 1 \) then \( \lim_{t \to -\infty} y_t = 0 \) for any \( b \). From (8), it follows that \( |m| < 1 \) equates to the condition \( |\theta_1| < 1 \) or equivalently that the root of the AR polynomial is greater than one which is the condition for stationarity derived above.

This is an intuitively appealing link, namely the AR process is stationary if the associated homogeneous linear difference equation is stable and it turns out that this link is valid in the general case as well.

(ii) The stationarity condition in the AR(2) model:

Consider the model

\[
v_t = \theta_1 v_{t-1} + \theta_2 v_{t-2} + w_t
\]

(9)

As with the AR(1), the condition for stationarity involves the roots of the AR polynomial which this time is \( \theta(L) = 1 - \theta_1 L - \theta_2 L^2 \) and is related to the stability of the associated second order linear homogeneous difference equation.

In this case, the linear homogeneous difference equation of interest is:

\[
\theta(L)y_t = y_t - \theta_1 y_{t-1} - \theta_2 y_{t-2} = 0
\]

(10)

The solution to (10) depends on the roots of the second order polynomial (known as the "characteristic polynomial"),

\[
m^2 - \theta_1 m - \theta_2 = 0
\]

(11)

Since this polynomial is a quadratic, its roots are given by:

\[
m = \frac{\theta_1 \pm \sqrt{\theta_1^2 + 4\theta_2}}{2}
\]

(12)

There are three possibilities to be considered:
if \( \theta_1^2 + 4\theta_2 > 0 \) then there are two distinct real roots;

- if \( \theta_1^2 + 4\theta_2 = 0 \) then there is one real root repeated twice;

- if \( \theta_1^2 + 4\theta_2 < 0 \) then there are two imaginary roots that form a complex conjugate pair.

The solution to (10) depends on the nature of the roots of (11). Specifically, it can be shown that:

- if there are two distinct real roots, \( m_1 \) and \( m_2 \) then the solution is:
  \[ y_t = b_1 m_1^t + b_2 m_2^t \]
  where \( b_1 \) and \( b_2 \) depend on the initial conditions;

- if there is a repeated real root, \( m \), then the solution is
  \[ y_t = (b_1 + b_2 t)m^t \]
  where \( b_1 \) and \( b_2 \) depend on the initial conditions;

- if there imaginary roots, \( m \) and its complex conjugate \( \bar{m} \), then the solution is:
  \[ y_t = b_a m^t + b\bar{m}^t \]
  where \( b_a \) is the complex conjugate of \( b \) and both depend on the initial conditions; or equivalently, if the roots are written in polar coordinates as \( r\{\cos(\theta) \pm i\sin(\theta)\} \) then
  \[ y_t = r^t g_1 \cos(t\theta) + g_2 \]
  where \( r = \sqrt{mm^*} \) and \( g_1, g_2 \) are constants reflecting the initial conditions.

The implications for the stability of (10) are as follows:

- if the roots of the characteristic polynomial, (11), are real then the difference equation is stable if the absolute value of the roots is less than one.

- if the roots of the characteristic polynomial, (11), are imaginary then the difference equation is stable if \(|\rho| < 1\).

As in the AR(1) case, the condition for stationarity of the AR(2) is identical to the condition for stability of the associated linear homogenous difference equation. Since we are interested in the AR polynomial, it is more convenient to express these conditions in terms of the roots of the AR polynomial. As in the AR(1) case, the roots of the AR polynomial are the inverses of the roots of the characteristic polynomial. In the case of real roots, the condition is simply that the roots of the AR polynomial are greater than one. In the case of imaginary roots to the AR polynomial, \( \rho \) and its complex conjugate \( \bar{\rho} \) say, the condition is that \( \bar{\rho} = \sqrt{mm^*} \) is greater than one. This is referred to as the requirement that the roots are outside the unit circle where the terminology derives from the polar coordinate representation of the complex roots. Since real roots can be thought of
as complex roots for which the coefficient on the imaginary part is zero, this terminology is used regardless of the nature of the roots.

(iii) Condition for stationarity in the AR(p) model

Let \( \{v_t\} \) be generated via the model

\[
\theta(L)v_t = w_t
\]

where \( \theta(L) = 1 - \sum_{i=1}^{p} \theta_i L^i \) and \( w_t \sim iid(0, \sigma^2) \). The model implies a well behaved MA(\( \infty \)) representation if the roots of \( \theta(\lambda) \) lie outside the unit circle. This condition is known as the stationarity condition.