Solutions to Practice Problem Set 1

1. Define $y_t = (v_t, v_{t-1}, \ldots, v_{t-n})$. By assumption \{v_t\} is covariance stationary and so: $E[y_t] = \mu_y$ where $\mu_y$ is a $(n + 1) \times 1$ vector whose elements are all $\mu$; $\text{Var}[y_t] = \Sigma_y$ where $\Sigma_y$ is a $(n + 1) \times (n + 1)$ matrix whose $(k, \ell)$ element is $\text{Cov}[v_{t-k+1}, v_{t-\ell+1}] = \gamma_{k-\ell}$. By assumption, $y_t \sim N(\mu_y, \Sigma)$ and so the pdf of $y_t$ is:

$$p(y; \mu_y, \Sigma_y) = (2\pi)^{-(n+1)/2} |\Sigma_y|^{-1/2} \exp \left\{-0.5(y - \mu_y)'\Sigma_y^{-1}(y - \mu_y)\right\}$$

Since this pdf only depends on $\mu_y$ and $\Sigma_y$ and each of these are independent of $t$, it follows that the pdf of $y_t = (v_t, v_{t-1}, \ldots, v_{t-n})$ is independent of $t$.

2. Note that by definition $\sigma_{1,2} = \sigma_{2,1}$. Using the properties of $v_t$ the autocovariance matrices are as follows:

$$\Gamma_1 = \begin{bmatrix}
E[v_{1,t}v_{1,t-1}] & E[v_{1,t}v_{2,t-1}] \\
E[v_{2,t}v_{1,t-1}] & E[v_{2,t}v_{2,t-1}]
\end{bmatrix} = \begin{bmatrix}
\theta_1 \sigma_{1,1} & \theta_1 \sigma_{1,2} \\
\theta_2 \sigma_{1,2} & \theta_2 \sigma_{2,2}
\end{bmatrix}$$

and

$$\Gamma_{-1} = \begin{bmatrix}
E[v_{1,t}v_{1,t+1}] & E[v_{1,t}v_{2,t+1}] \\
E[v_{2,t}v_{1,t+1}] & E[v_{2,t}v_{2,t+1}]
\end{bmatrix} = \begin{bmatrix}
\theta_1 \sigma_{1,1} & \theta_2 \sigma_{1,2} \\
\theta_1 \sigma_{1,2} & \theta_2 \sigma_{2,2}
\end{bmatrix}$$

From these two equations it can be seen that $\Gamma_1 \neq \Gamma_{-1}$ (providing $\theta_1 \neq \theta_2$) but $\Gamma_1 = \Gamma_{-1}$.

3. If $\sum_{i=0}^{\infty} |\psi_i| < \infty$ then there exists $N < \infty$ such that $|\psi_j| < 1$ for all $j \geq N$. Therefore, for $j \geq N$ it follows that $\psi_j^2 < |\psi_j|$ and so

$$\sum_{i=0}^{\infty} \psi_i^2 = \sum_{i=0}^{N-1} \psi_i^2 + \sum_{i=N}^{\infty} \psi_i^2 < \sum_{i=0}^{N-1} \psi_i^2 + \sum_{i=N}^{\infty} |\psi_i|$$

(1)

Now absolute summability implies $|\psi_i| < \infty$ for all $i$ and so, since $N$ is finite, it follows that $\sum_{i=0}^{N-1} \psi_i^2$ is finite. Absolute summability also implies that $\sum_{i=N}^{\infty} |\psi_i|$ is finite. Therefore, it follows from (1) and $\sum_{i=0}^{\infty} |\psi_i| < \infty$ that $\sum_{i=0}^{\infty} \psi_i^2 < \infty$.
4. Given the form of the autocovariances, it follows that

\[ |\gamma_j| = |\sigma^2 \sum_{k=0}^{\infty} \psi_{j+k} \psi_k| \leq \sigma^2 \sum_{k=0}^{\infty} |\psi_{j+k} \psi_k| \]  

(2)

and so

\[ \sum_{j=0}^{\infty} |\gamma_j| \leq \sigma^2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\psi_{j+k} \psi_k| \]  

(3)

Now, we have

\[ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\psi_{j+k} \psi_k| = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\psi_{j+k} \psi_k| = \sum_{k=0}^{\infty} |\psi_k| \sum_{j=0}^{\infty} |\psi_{j+k}| \]  

(4)

\( \sum_{i=0}^{\infty} |\psi_i| < \infty \) implies there exists \( b < \infty \) such that \( \sum_{i=0}^{\infty} |\psi_i| < b \) and so \( \sum_{j=0}^{\infty} |\psi_{j+k}| < b \) for all \( k = 0, 1, 2, \ldots \). Therefore,

\[ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\psi_{j+k} \psi_k| \leq \sum_{k=0}^{\infty} |\psi_k| b \leq b^2 < \infty \]  

(5)

Combining (3)-(5), we obtain \( \sum_{i=0}^{\infty} |\gamma_i| \leq \sigma^2 b^2 < \infty \).

5. Put \( \Theta(L) = 1 - \theta_1 L - \theta_2 L^2 \).

6. This model for \( y_t \) can be written as \( \Theta(L)y_t = \epsilon_t \) where \( \Theta(L) = 1 - 0.25L^2 \). Now, it is easily verified by multiplying out that \( \Theta(L) = (1 - 0.5L)(1 + 0.5L) \).

7. We have

\[ A(L)y_t = (1 + 0.9L + 0.8L^2)y_t \]

\[ = y_t + 0.9y_{t-1} + 0.8y_{t-2} \]

\[ = (x_t + v_t) + 0.9(x_{t-1} + v_{t-1}) + 0.8(x_{t-2} + v_{t-2}) \]

\[ = x_t + 0.9x_{t-1} + 0.8x_{t-2} + v_t + 0.9v_{t-1} + 0.8v_{t-2} \]

\[ = A(L)x_t + A(L)v_t \]