Solutions to Assessed Problem Set 1

1(i) The AR(2) in equation (1) of the question implies that

\[ v_t - \theta_1 v_{t-1} - \theta_2 v_{t-2} = w_t \]  \hspace{1cm} (1)

From the $MA(\infty)$ representation it follows that

\[ v_t = \psi_0 w_t + \psi_1 w_{t-1} + \psi_2 w_{t-2} + \psi_3 w_{t-3} + \ldots \]  \hspace{1cm} (2)
\[ v_{t-1} = \psi_0 w_{t-1} + \psi_1 w_{t-2} + \psi_2 w_{t-3} + \psi_3 w_{t-4} + \ldots \]  \hspace{1cm} (3)
\[ v_{t-2} = \psi_0 w_{t-2} + \psi_1 w_{t-3} + \psi_2 w_{t-4} + \psi_3 w_{t-5} + \ldots \]  \hspace{1cm} (4)

If we substitute (2)-(4) into the left hand side (LHS) of (1) and rearrange so that each $w_{t-i}$ appears only once then the equality can only hold if: (a) the resulting coefficient on $w_t$ on the LHS of (1) is unity; (b) the resulting coefficients on $w_{t-i}$ on the LHS of (1) are all zero for $i > 0$. This logic implies that:

\[ \psi_0 = 1 \]  \hspace{1cm} (5)
\[ \psi_1 - \theta_1 \psi_0 = 0 \]  \hspace{1cm} (6)
\[ \psi_j - \theta_1 \psi_{j-1} - \theta_2 \psi_{j-2} = 0, \quad \text{for} \quad j = 2, 3, \ldots \]  \hspace{1cm} (7)

1(ii)(a) Using the “Handout on the Stationarity Condition” (HSC), we know that if the characteristic polynomial of the difference equation has distinct roots then

\[ \psi_j = b_1 m_1^j + b_2 m_2^j \]  \hspace{1cm} (8)

where $b_1$ and $b_2$ depend on the initial conditions (and $m_1$ and $m_2$ are complex conjugates if the roots are imaginary). Setting $j = 0$, it follows from (5) and (8) that

\[ b_1 + b_2 = 1 \]  \hspace{1cm} (9)

Setting $j = 1$, it follows from (6) and (8) that

\[ b_1 m_1 + b_2 m_2 - \theta_1 (b_1 + b_2) = 0 \]  \hspace{1cm} (10)

Equation (9) implies $b_1 = 1 - b_2$, and substituting this expression into (10), we obtain

\[ (1 - b_2) m_1 + b_2 m_2 - \theta_1 = 0 \]  \hspace{1cm} (11)

which yields

\[ b_2 = \frac{\theta_1 - m_1}{m_2 - m_1} \]  \hspace{1cm} (12)
Now the characteristic polynomial must satisfy

\[ m^2 - \theta_1 m - \theta_2 = 0 = (m - m_1)(m - m_2) \quad (13) \]

and so multiplying out the right hand equation in (13), it must follow that \( \theta_1 = m_1 + m_2 \). Substituting this value into (12), we obtain

\[ b_2 = \frac{m_2}{m_2 - m_1} \quad (14) \]

Substituting for \( b_2 \) in (9) gives

\[ b_1 = \frac{m_1}{m_1 - m_2} \quad (15) \]

Equations (8), (14) and (15) give the desired result.

1.(ii)(b) Using HSC, we know that if the characteristic polynomial of the difference equation has repeated roots then

\[ \psi_j = (b_1 + b_2j)m_a^j \quad (16) \]

where \( m_a \) is the root, \( b_1 \) and \( b_2 \) depend on the initial conditions. Setting \( j = 0 \), it follows from (5) and (16) that

\[ b_1 = 1 \quad (17) \]

Setting \( j = 1 \), it follows from (6), (16) and (17) that

\[ (1 + b_2)m_a = \theta_1 \quad (18) \]

Since \( m^2 - \theta_1 m - \theta_2 = 0 = (m - m_a)^2 \), it must follow that \( \theta_1 = 2m_a \). Substituting for \( \theta_1 \) in (18) yields \( b_2 = 1 \). Finally, substituting \( b_1 = b_2 = 1 \) into (16) gives the desired result.

(iii) Consider first the case of distinct roots. From (8), it follows that

\[ \sum_{j=0}^{\infty} |\psi_j| = \sum_{j=0}^{\infty} |b_1 m_1^j + b_2 m_2^j| \quad (19) \]

Now

\[
|b_1 m_1^j + b_2 m_2^j| \leq |b_1 m_1^j| + |b_2 m_2^j| \\
\leq |b_1||m_1^j| + |b_2||m_2^j| \\
\leq |b_1||m_1|^j + |b_2||m_2|^j \\
\leq 2b_a m_a^j \quad (20)
\]

where \( b_a = max\{|b_1|, |b_2|\}, \ m_a = max\{|m_1|, |m_2|\} \). Using (20) and (19), it follows that

\[ \sum_{j=0}^{\infty} |\psi_j| \leq 2b_a \sum_{j=0}^{\infty} m_a^j \quad (21) \]
Now, \( m_a < 1 \) (because the roots of the characteristic are inside the unit circle) and so using Practice Problem Set #2 Question 1, (21) can be rewritten as
\[
\sum_{j=0}^{\infty} |\psi_j| \leq \frac{2b_a}{1 - m_a} < \infty \tag{22}
\]
where the last inequality follows because \( m_a < 1 \) which in turn implies \( b_a < \infty \) via (14)-(15).

Now consider the case of repeated roots. In this case, we have
\[
\sum_{j=0}^{\infty} |\psi_j| = \sum_{j=0}^{\infty} |(1 + j)m^j_s| \\
\leq \sum_{j=0}^{\infty} (1 + j)|m_s|^j \\
\leq 2 \sum_{j=0}^{\infty} j|m_s|^j \\
= \frac{2m_s}{(1 - m_s)^2} \tag{23}
\]
where the last equality follows from the hint given in the question and \( |m_s| < 1 \). It follows from (23) and \( |m_s| < 1 \) that \( \sum_{j=0}^{\infty} |\psi_j| < \infty \).

2(a) Practice Problem Set #2 Question 5, we know that \( \gamma_0 = \sigma^2(1 + \phi^2) \), \( \gamma_1 = \sigma^2\phi \) and \( \gamma_i = 0 \) for \( i > 1 \). Therefore, it follows from the definition of \( S \) that for this model the long run variance is:
\[
S = \gamma_0 + 2\gamma_1 \\
= \sigma^2(1 + \phi^2) + 2\sigma^2\phi \\
= \sigma^2(1 + \phi^2 + 2\phi) \\
= \sigma^2(1 + \phi)^2 \\
= \sigma^2\phi(1)^2
\]

2(b) Practice Problem Set #2 Question 3(b), we know that \( \gamma_0 = \sigma^2/(1 - \theta^2) \) and \( \gamma_i = \theta \gamma_0 \) for \( i > 0 \). Therefore, it follows from the definition of \( S \) that for this model the long run variance is:
\[
S = \gamma_0 + 2 \sum_{i=1}^{\infty} \theta^i \gamma_0 \\
= \frac{\sigma^2}{1 - \theta^2} \left[ 1 + 2 \sum_{i=1}^{\infty} \theta^i \right]
\]
We first need to determine the autocovariances of an ARMA(1,1) process, denoted \( \{ \gamma_i \} \). To this end, it is convenient to write the process as follows:

\[
\begin{align*}
v_t &= \theta v_{t-1} + \epsilon_t \\
\epsilon_t &= w_t + \phi w_{t-1}
\end{align*}
\]

(24)

(25)

From (24), it follows that

\[
\begin{align*}
\gamma_0 &= E[v_t^2] \\
&= E[(\theta v_{t-1} + \epsilon_t)^2] \\
&= E[(\theta v_{t-1})^2] + 2 Cov[\theta v_{t-1}, \epsilon_t] + Var[\epsilon_t]
\end{align*}
\]

(26)

We now evaluate each term on the right hand side of (26). Since \( E[(\theta v_{t-1})^2] = \theta^2 E[v_{t-1}^2] \) and \( \{ v_t \} \) is covariance stationary, it follows that \( E[(\theta v_{t-1})^2] = \theta^2 \gamma_0 \). For \( Cov[\theta v_{t-1}, \epsilon_t] \), we first note that \( |\theta| < 1 \) and (24) together imply that \( v_t = \epsilon_t + \sum_{i=1}^{\infty} \psi_i \epsilon_{t-i} \) and hence that

\[
v_{t-1} = \epsilon_{t-1} + \sum_{i=1}^{\infty} \psi_i \epsilon_{t-i}
\]

Now \( \epsilon_t \sim MA(1) \) and so

\[
Cov[v_{t-1}, \epsilon_t] = Cov[\epsilon_{t-1} + \sum_{i=1}^{\infty} \psi_i \epsilon_{t-1-i}, \epsilon_t] = Cov[\epsilon_{t-1}, \epsilon_t]
\]

because \( Cov[\epsilon_t, \epsilon_{t-k}] = 0 \) for \( k > 1 \). Now, using (25) we have

\[
Cov[\epsilon_{t-1}, \epsilon_t] = Cov[w_t + \phi w_{t-1}, \epsilon_t] = \phi \sigma^2
\]

because \( w_t \) is white noise. Therefore, we have \( Cov[\theta v_{t-1}, \epsilon_t] = \theta \phi \sigma^2 \). Finally, since \( \epsilon_t \sim MA(1) \), it follows that \( Var[\epsilon_t] = \sigma^2 (1 + \phi^2) \). Substituting the appropriate expressions into the right hand side of (26), we obtain

\[
\gamma_0 = \theta^2 \gamma_0 + 2 \theta \phi \sigma^2 + \sigma^2 (1 + \phi^2)
\]

(27)

With rearrangement, (27) yields

\[
\gamma_0 = \frac{\sigma^2 (1 + 2 \theta \phi + \phi^2)}{1 - \theta^2}
\]

(28)
For $\gamma_1$, we note that (24) implies

$$
\begin{align*}
\gamma_1 & = E[v_t, v_{t-1}] \\
& = E[v_{t-1}(\theta v_{t-1} + \epsilon_t)] \\
& = \theta E[v_{t-1}^2] + \text{Cov}[v_{t-1}, \epsilon_t] \\
& = \theta \gamma_0 + \phi \sigma^2
\end{align*}
$$

(29)

For $\gamma_j$, $j > 1$ we note that (24) implies

$$
\begin{align*}
\gamma_j & = E[v_t, v_{t-j}] \\
& = E[v_{t-j}(\theta v_{t-1} + \epsilon_t)] \\
& = \theta E[v_{t-1}v_{t-j}] + \text{Cov}[v_{t-j}, \epsilon_t] \\
& = \theta \gamma_{j-1} \\
& = \theta^{j-1} \gamma_1 \quad \text{(after recursive backward substitution)} \\
& = \theta^{j} \gamma_0 + \theta^{j-1} \phi \sigma^2
\end{align*}
$$

(30)

where the last identity uses (29) and the fourth identity uses the fact that $\epsilon_t \sim MA(1)$ and so is uncorrelated with $\{e_{t-2}, e_{t-3}, e_{t-4}, \ldots\}$ and hence with $v_{t-j} = e_{t-j} + \sum_{i=1}^{\infty} \psi_j e_{t-j-i}$ for $j > 1$.

Now consider $S$. By definition of the long run variance, we have

$$
S = \gamma_0 + 2 \sum_{i=1}^{\infty} \gamma_i
$$

$$
= \gamma_0 + 2 \sum_{i=1}^{\infty} \{ \theta^i \gamma_0 + \theta^{i-1} \phi \sigma^2 \}
$$

$$
= \gamma_0 + 2 \theta \gamma_0 \sum_{i=0}^{\infty} \theta^i + 2 \phi \sigma^2 \sum_{i=0}^{\infty} \theta^i
$$

$$
= \gamma_0 + 2 \theta \gamma_0 \left\{ \frac{\theta}{1-\theta} \right\} + 2 \left\{ \frac{\phi \sigma^2}{1-\theta} \right\}
$$

$$
= \left\{ \frac{1+\theta}{1-\theta} \right\} \gamma_0 + 2 \left\{ \frac{\phi \sigma^2}{1-\theta} \right\}
$$

(31)

Using (28), it follows that

$$
\left\{ \frac{1+\theta}{1-\theta} \right\} \gamma_0 = \left\{ \frac{1+\theta}{1-\theta} \right\} \left\{ \frac{\sigma^2(1 + 2\theta \phi + \phi^2)}{1-\theta^2} \right\}
$$

$$
= \frac{\sigma^2(1 + 2\theta \phi + \phi^2)}{(1-\theta)^2}
$$

(32)
Substituting (32) into (31), we obtain

\[ S = \frac{\sigma^2 (1 + 2\theta \phi + \phi^2)}{(1 - \theta)^2} + 2 \left\{ \frac{\phi \sigma^2}{1 - \theta} \right\} \]

\[ = \frac{\sigma^2 \{ 1 + 2\theta \phi + \phi^2 + 2\phi (1 - \theta) \}}{(1 - \theta)^2} \]

\[ = \frac{\sigma^2 (1 + \phi)^2}{1 - \theta)^2} \]

\[ = \frac{\sigma^2 \phi(1)^2}{\theta(1)^2} \]

which is the desired result.